



Πανεπιστήμιο Μακεδονίας

Τ.Ε.Ι. Δυτικής Μακεδονίας

Π.Μ.Σ Εφαρμοσμένης Πληροφορικής

Διπλωματική Εργασία

Θέμα

Εφαρμοσμένη Θεωρία Πινάκων

Επιβλέπον Καθηγητής

Πετράκης Ανδρέας

Μεταπτυχιακός Φοιτητής

Τσαγκαρλή Αθηνά

Κοζάνη , 2011

ΠΕΡΙΛΗΨΗ

Στα μαθηματικά ονομάζουμε πίνακα ή μήτρα ν γραμμών και μ στηλών μία ορθογώνια διάταξη σε σχήμα ορθογώνιου παραλληλογράμμου που περιέχει νχμ πλήθος στοιχείων. Οι εγγραφές ή στοιχεία του πίνακα μπορούν να είναι αριθμοί ή, πιο γενικά, οποιεσδήποτε αφηρημένες ποσότητες τις οποίες μπορούμε να προσθέσουμε και να πολλαπλασιάσουμε.

Οι πίνακες είναι ένα βασικό εργαλείο για πολλές περιοχές των Μαθηματικών , αναμφισβήτητη είναι όμως η χρησιμότητά του στις εφαρμογές και σε άλλες μεθόδους, σε επιστήμες όπως η Φυσική και η Οικονομία καθώς και η δύναμη τους σαν ερευνητικό εργαλείο.

Στην παρούσα διπλωματική αφού αρχικά δώσαμε κάποιους ορισμούς σχετικούς με την Θεωρία Πινάκων , ασχοληθήκαμε με την Τριγωνομετρία Πινάκων. Ορίσαμε το Ημίτονο , το Συνημίτονο , την Εφαπτομένη και τη Συνεφαπτομένη ενός 2×2 αλλά και ενός 3×3 πίνακα και δείξαμε ότι οι Τριγωνομετρικές Ταυτότητες ισχύουν και για τους Τριγωνομετρικούς Πίνακες παρουσιάζοντας και κάποια παραδείγματα.

Περιεχόμενα

ΠΕΡΙΛΗΨΗ	2
1. Αριθμητική Παρεμβολή.....	4
1.1 Ορισμοί.....	4
1.2 Υπαρξη και μοναδικότητα του πολυωνύμου παρεμβολής.....	5
1.3 Διαιρεμένες διαφορές.....	8
1.4 Πολυώνυμο παρεμβολής Newton	9
1.5 Πολυώνυμο παρεμβολής Hermite	13
1.6 Σφάλμα παρεμβολής.....	17
2. Ιδιοτιμές και Ιδιοδιανύσματα πίνακα	18
2.1. Ιδιοτιμές και Ιδιοδιανύσματα	18
2.2. Διαγωνιοποίηση πινάκων	19
2.3. Γεωμετρική Πολλαπλότητα	20
2.4. Χαρακτηριστικό Πολυώνυμο.....	20
1.5. Αλγεβρική Πολλαπλότητα	21
3. Ρίζες πολυωνύμου, βαθμός πολλαπλότητας με τη χρήση παραγώγων.....	25
4. «Τριγωνομετρία Πινάκων».....	30
4.1 Αριθμητική παρεμβολή	30
4.1.1.Ορισμός πίνακα 2×2	31
4.1.2. Ορισμός πίνακα 3×3	34
4.1.3. Ορισμός Πίνακα $N \times N$	63
4.2 Με δυναμοσειρές.....	65
4.3 Τριγωνομετρικές Ταυτότητες	67
4.4 Παραδείγματα	68
ΒΙΒΛΙΟΓΡΑΦΙΑ	120

1. Αριθμητική Παρεμβολή

1.1 Ορισμοί

Η **παρεμβολή** αποτελεί μια από τις πλέον διαδεδομένες προσεγγιστικές τεχνικές στον τομέα της αριθμητικής ανάλυσης και των υπολογιστικών μαθηματικών. Με τον όρο παρεμβολή εννοούμε το πρόβλημα της προσέγγισης μιας συνάρτησης f της οποίας είναι γνωστές οι τιμές σε διακεκριμένα σημεία $x_i, i = 0, 1, 2, \dots, n$, από μια άλλη συνάρτηση g πιο εύχρηστη. Όταν η προσεγγιστική συνάρτηση που χρησιμοποιείται είναι πολυώνυμο τότε η μέθοδος καλείται «**πολυωνυμική παρεμβολή**».

Το κλασσικό πρόβλημα παρεμβολής με πολυώνυμα διατυπώνεται όπως πιο κάτω:

Πρόβλημα:

Ας είναι $n+1$ διακεκριμένα σημεία x_0, x_1, \dots, x_n στα οποία οι τιμές $f(x_0), f(x_1), \dots, f(x_n)$, της συνάρτησης f είναι γνωστές. Να βρεθεί πολυώνυμο $p_n(x)$, βαθμού n , το οποίο να εμφανίζει τις ίδιες τιμές με την f στα ίδια $n+1$ σημεία.

Δηλαδή ψάχνουμε ένα πολυώνυμο $p_n(x)$ τέτοιο ώστε να ικανοποιεί τις πιο κάτω «**συνθήκες παρεμβολής**»

$$p_n(x_i) = f(x_i), \quad \text{για } i = 0, 1, \dots, n \quad (1.1)$$

Τα σημεία x_0, x_1, \dots, x_n καλούνται «**σημεία παρεμβολής**» ή «**κόμβοι παρεμβολής**» ενώ το $p_n(x)$ «**πολυώνυμο παρεμβολής**» βαθμού n .

1.2 Ύπαρξη και μοναδικότητα του πολυωνύμου παρεμβολής

Θεώρημα 1.1: Για οποιοδήποτε σύνολο $n+1$ διακεκριμένων σημείων x_0, x_1, \dots, x_n και των αντίστοιχων τιμών της συνάρτησης f υπάρχει μοναδικό πολυώνυμο $p(x) \in P_n$ τέτοιο ώστε

$$f(x_i) = p(x_i) \text{ για } i = 0, 1, \dots, n$$

Απόδειξη:

Θα δώσουμε δύο τρόπους απόδειξης. Στον δεύτερο τρόπο η ύπαρξη και η μοναδικότητα του πολυωνύμου παρεμβολής χωρίζεται σε δύο σκέλη.

1ος τρόπος: Ένα πολυώνυμο $p(x) = \sum_{i=0}^n a_i x^i$ ικανοποιεί τις συνθήκες παρεμβολής, σχέση (1.1), αν-ν οι συντελεστές a_i αποτελούν λύση του γραμμικού συστήματος

$$a_0 + a_1 x_0 + \dots + a_n x_0^n = f(x_0)$$

$$a_0 + a_1 x_1 + \dots + a_n x_1^n = f(x_1)$$

.....

$$a_0 + a_1 x_n + \dots + a_n x_n^n = f(x_n)$$

Αυτό το σύστημα έχει μοναδική λύση αν-ν η ορίζουσα του δεν μηδενίζεται. Μια τέτοια ορίζουσα καλείται **ορίζουσα του Vandermonde** και συμβολίζεται με $VDM(x_0, x_1, \dots, x_n)$. Έτσι

$$VDM(x_0, x_1, \dots, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} & x_1^n \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & x_n^n \end{vmatrix}$$

$$\text{με } VDM(x_0, x_1, \dots, x_n) = \prod_{0 \leq i < j \leq n} (x_i - x_j).$$

Αφού τα σημεία παρεμβολής είναι διακεκριμένα, δηλαδή $x_i \neq x_j$ για $i \neq j$, τότε η ορίζουσα δεν μηδενίζεται και το πρόβλημα έχει μια και μοναδική λύση.

2^{ος} τρόπος:

Έγγρη: ας είναι $\{l_i\}_{i=0}^n$ μια βάση του χώρου P_n των πολυωνύμων βαθμού n . Τότε το $p(x) \in P_n$ μπορεί να γραφεί σαν γραμμικός συνδυασμός των στοιχείων της βάσης, δηλαδή να παρασταθεί ως

$$p(x) = \sum_{i=0}^n b_i l_i(x) \quad (1.2)$$

με την ιδιότητα

$$p(x_i) = \sum_{j=0}^n b_j l_j(x_i) = f(x_i), i = 0, 1, \dots, n \quad (1.3)$$

Αν ορίσουμε τα

$$l_i \in P_n : l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}, i = 0, 1, \dots, n \quad (1.4)$$

τότε

$$l_i(x_j) = \delta_{i,j} := \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

οπότε από την (1.3) προκύπτει ότι

$$b_i = f(x_i)$$

Άρα το πολυώνυμο παρεμβολής υπάρχει και έχει την μορφή

$$p(x) = \sum_{i=0}^n f(x_i)l_i(x)$$

Μοναδικότητα: υποθέτουμε ότι υπάρχει ακόμα ένα πολυώνυμο παρεμβολής το $q(x) \in P_n$ που παρεμβάλλει την f στα $n+1$ διακεκριμένα σημεία x_0, x_1, \dots, x_n και άρα ικανοποιεί τις πιο κάτω συνθήκες παρεμβολής

$$q_n(x_i) = f(x_i), i = 0, 1, \dots, n \quad (1.5)$$

Έστω

$$r_n(x) = p_n(x) - q_n(x) \quad (1.6)$$

Από τι σχέσεις (1.1),(1.2),(1.3) έπειται ότι

$$r_n(x_i) = p_n(x_i) - q_n(x_i) = 0, i = 0, 1, \dots, n$$

Οπότε το πολυώνυμο $r_n(x)$ που ανήκει στο P_n και έχει $n+1$ διακεκριμένες ρίζες θα πρέπει να είναι ταυτοτικά μηδέν. Άρα από την (1.6) προκύπτει ότι

$$r_n(x) = p_n(x) - q_n(x) = 0 \Rightarrow p_n(x) \equiv q_n(x),$$

που επιβεβαιώνει ότι το πολυώνυμο παρεμβολής είναι μοναδικό. \square

1.3 Διαιρεμένες διαφορές

Διαιρεμένη διαφορά πρώτης τάξης στα σημεία x_0, x_1 καλείται το πηλίκο

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \text{ και συμβολίζεται με } f[x_0, x_1]. \text{ Δηλαδή}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (1.7)$$

Σαν διαιρεμένη διαφορά 2^{ης} τάξης στα σημεία x_0, x_1, x_2 ορίζεται

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \quad (1.8)$$

ενώ σαν διαιρεμένη διαφορά τάξης n στα σημεία x_0, x_1, \dots, x_n ορίζεται

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0} \quad (1.9)$$

Ο πίνακας διαιρεμένων διαφορών είναι:

x	f	$f[,]$	$f[,,]$	\dots	$f[,,\dots,]$
x_0	$f(x_0)$				
		$f[x_0, x_1]$			
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$		
			$f[x_1, x_2]$		
x_2	$f(x_2)$				
				$f[x_0, x_1, \dots, x_n]$	
.	
			$f[x_{n-2}, x_{n-1}, x_n]$		
			$f[x_{n-1}, x_n]$		
x_n	$f(x_n)$				

1.4 Πολυώνυμο παρεμβολής Newton

Ένα πολυώνυμο στη μορφή Newton με κέντρα τα σημεία παρεμβολής x_0, x_1, \dots, x_n γράφεται:

$$p_n(x) = \alpha_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1) + \dots + \alpha_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) = \\ = \sum_{i=0}^n \alpha_i \prod_{j=0}^{i-1} (x - x_j)$$
(1.10)

Είναι εύκολο να δειχθεί από τις συνθήκες παρεμβολής $p_n(x_i) = f(x_i)$, για $i = 0, 1, \dots, n$ ότι:

$$p_n(x_0) = f(x_0) = \alpha_0$$
(1.11)

$$p_n(x_1) = f(x_1) = \alpha_0 + \alpha_1(x_1 - x_0)$$

$$p_n(x_2) = f(x_2) = \alpha_0 + \alpha_1(x_2 - x_0) + \alpha_2(x_2 - x_0)(x_2 - x_1)$$

.....

$$p_n(x_k) = f(x_k) = \sum_{i=0}^k \alpha_i \prod_{j=0}^{i-1} (x_k - x_j), k = 0, 1, 2, \dots, n$$
(1.12)

Οι εξισώσεις (1.11) μπορούν να λυθούν για τα α_i στη σειρά αρχίζοντας με α_0 . Ετσι, το α_0 εξαρτάται από το $f(x_0)$, το α_1 από το $f(x_0)$ και $f(x_1)$ κ.ο.κ. Γενικότερα το α_n εξαρτάται από τα $f(x_0), f(x_1), \dots, f(x_n)$. Με άλλα λόγια α_n εξαρτάται από την f στα σημεία x_0, \dots, x_n .

Ο συμβολισμός του είναι :

$$a_n = f[x_0, x_1, \dots, x_n]$$
(1.13)

Χρησιμοποιώντας τις (1.11) βρίσκουμε τους τύπους:

$$a_0 = f[x_0] = f(x_0)$$
(1.14)

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Μπορούμε να συνεχίσουμε έτσι, αλλά προτιμούμε τον ακόλουθο αναδρομικό τύπο:

$$a_k = f[x_0, x_1, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}, k = 1, 2, \dots, n \quad (1.15)$$

που είναι η διαιρεμένη διαφορά k -τάξης στα σημεία x_0, x_1, \dots, x_k .

Δικαιολογούμε αυτόν τον τύπο ως ακολούθως:

Η $f[x_0, x_1, \dots, x_k]$ είναι ο συντελεστής του x^k στο πολυώνυμο p_k βαθμού $\leq k$ που παρεμβάλλει η f στα x_0, x_1, \dots, x_k . Μία ματιά στο δεξιό μέλος της (1.15) δείχνει ότι δύο άλλα πολυώνυμα q και r περιβάλλουν την f , το q παρεμβάλλει την f στα x_0, \dots, x_{k-1} και το r παρεμβάλλει την f στα x_1, \dots, x_k βαθμού $\leq k-1$. Η σχέση μεταξύ των πολυωνύμων $p_k(x), q_{k-1}(x), r_{k-1}(x)$ είναι:

$$p_k(x) = r_{k-1}(x) + \frac{x - x_k}{x_k - x_0} [r_{k-1}(x) - q_{k-1}(x)] \quad (1.16)$$

Για να αποδείξουμε την (1.16) παρατηρούμε ότι το δεξιό μέλος είναι ένα πολυώνυμο βαθμού $\leq k$. Υπολογίζοντάς το στο x_0 δίνει $f(x_0)$:

$$r(x_0) + \frac{x - x_k}{x_k - x_0} [r(x_0) - q(x_0)] = q(x_0) = f(x_0)$$

Υπολογίζοντάς το στα x_i ($i = 1, 2, \dots, k-1$) το αριστερό μέλος της (1.16) είναι $f(x_i)$ και το δεξιό είναι:

$$r(x_i) + \frac{x_i - x_k}{x_k - x_0} [r(x_i) - q(x_i)] = f(x_i) + \frac{x_i - x_k}{x_k - x_0} [f(x_i) - f(x_i)] = f(x_i)$$

Όμοια στο x_k παίρνουμε $f(x_k)$:

$$r(x_k) + \frac{x_k - x_k}{x_k - x_0} [r(x_k) - q(x_k)] = r(x_k) = f(x_k)$$

Από τη μοναδικότητα του πολυωνύμου παρεμβολής, το δεξιό μέλος της (1.16) πρέπει να είναι $p_k(x)$ και η εξίσωση (1.16) αποδείχθηκε. Εξισώνοντας τους συντελεστές του x^k στα δύο μέλη της (1.16) παίρνουμε την (1.15). Πράγματι

$f[x_1, \dots, x_k]$ είναι ο συντελεστής του x^{k-1} στο $r(x)$ και $f[x_0, \dots, x_{k-1}]$ ο συντελεστής του x^{k-1} στο $q(x)$.

Επειδή τα x_0, x_1, \dots, x_k και κ είναι αυθαίρετα, ο αναδρομικός τύπος (1.15) μπορεί να γραφεί:

$$f[x_i, x_{i+1}, \dots, x_{j-1}, x_j] = \frac{f[x_{i+1}, \dots, x_j] - f[x_i, \dots, x_{j-1}]}{x_j - x_i} \quad (1.17)$$

Οι συντελεστές $\alpha_0, \alpha_1, \dots, \alpha_n$ στον τύπο (1.10) έχουν προσδιοριστεί από τις (1.15) σαν διαιρεμένες διαφορές ώστε να ικανοποιούνται οι συνθήκες παρεμβολής.

Το πολυώνυμο που προκύπτει είναι το εξής:

$$\begin{aligned} p_n(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots + \\ &\quad + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1}) \end{aligned} \quad (1.18)$$

δηλαδή,

$$p_n(x) = \sum_{i=0}^n f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j) \quad (1.19)$$

και καλείται πολυώνυμο παρεμβολής Newton.

Σημειώνουμε ότι αν υποθέσουμε ότι x_1 συγκλίνει στο x_0 τότε

$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ συγκλίνει στην $f'(x_0)$ από τον ορισμό της παραγώγου. Δηλαδή $f[x_0, x_0] = f'(x_0)$ και γενικότερα

$$f[x_i, x_i] = f'(x_i) \quad (1.20)$$

$$\text{ενώ αν και } x_2 \text{ συγκλίνει στο } x_0 \text{ τότε } f[x_0, x_0, x_0] = \frac{f''(x_0)}{2!}$$

Γενικά αν έχουμε τα σημεία $x_0 \leq x_1 \leq \dots \leq x_n$, από τα οποία μερικά μπορεί να είναι τα ίδια, δηλαδή $x_i = x_{i+1} = \dots = x_{i+k}$, τότε η πεπερασμένη διαφορά τάξης k δίνεται

$$f[x_i, \dots, x_{i+k}] = \frac{f^{(k)}(x_i)}{k!} \quad (1.21).$$

1.5 Πολυώνυμο παρεμβολής Hermite

Σκοπός της παρεμβολής Hermite είναι ο προσδιορισμός ενός πολυωνύμου το οποίο να συμφωνεί όχι μόνο με τις τιμές μιας συνάρτησης σε διακεκριμένα σημεία αλλά και με τις τιμές των παραγώγων αυτής στα ίδια σημεία. Δηλαδή ένα πρόβλημα παρεμβολής Hermite επιπρόσθετα με τις τιμές της συνάρτησης f στα σημεία x_0, x_1, \dots, x_n διαχειρίζεται και τις τιμές της $f^{(k)}$ στα x_0, x_1, \dots, x_n . Για $f \in C^M([a, b])$, δηλαδή η συνάρτηση έχει συνεχείς παραγώγους μέχρι M τάξης στο διάστημα $[a, b]$, οι συνθήκες παρεμβολής είναι:

$$p^{(k)}(x_i) = f^{(k)}(x_i) \quad (1.22)$$

για $i = 0, 1, \dots, n$ και $k = 0, 1, \dots, m$ όπου $m \leq M$.

Η παρεμβολή Hermite, μπορεί να θεωρηθεί σαν παρεμβολή σε πολλαπλά σημεία. Οπότε μπορούμε να χρησιμοποιήσουμε τον πίνακα των διαιρεμένων διαφορών και το πολυώνυμο του Newton. Έστω ότι έχουμε τα σημεία $x_0 \leq x_1$, στα οποία μας είναι γνωστές οι τιμές της συνάρτησης f και της παραγώγου αυτής, f' . Τότε μπορούμε να επιτύχουμε παρεμβολή Hermite στα σημεία x_0, x_1 θεωρώντας το καθένα βαθμού πολλαπλότητας 2. Ο πίνακας διαιρεμένων διαφορών γίνεται:

$p(x_0) = f(x_0)$	x_0	$f(x_0)$			
			$f'(x_0)$		
$p'(x_0) = f'(x_0)$	x_0	$f(x_0)$		$f[x_0, x_0, x_1]$	
			$f[x_0, x_1]$		$f[x_0, x_0, x_1, x_1]$
$p(x_1) = f(x_1)$	x_1	$f(x_1)$		$f[x_0, x_1, x_1]$	
			$f'(x_1)$		
$p'(x_1) = f'(x_1)$	x_1	$f(x_1)$			

Τότε το πολυώνυμο παρεμβολής γίνεται :

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 \\ + f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1)$$

ή λόγω της μοναδικότητας του πολυωνύμου παρεμβολής :

$$p(x) = f(x_1) + f'(x_1)(x - x_1) + f[x_0, x_1, x_1](x - x_1)^2 \\ + f[x_0, x_0, x_1, x_1](x - x_1)^2(x - x_0)$$

Γενικά, το πολυώνυμο παρεμβολής Hermite που παρεμβάλλει την f και την f' σε $n+1$ σημεία x_0, x_1, \dots, x_n είναι βαθμού $(2n+1)$. Για να παράγουμε το πολυώνυμο παρεμβολής Hermite, που ορίζεται στα πιο πάνω σημεία, χρησιμοποιούμε την σχέση

$$z_{2i} = z_{2i+1} = x_i, \text{ για } i = 0, 1, \dots, n \quad (1.23)$$

και ορίζουμε μια νέα ακολουθία σημείων

$$z_0, z_i, \dots, z_{2n+1} \quad (1.24)$$

Κατασκευάζουμε τον πίνακα των διαιρεμένων διαφορών με τα νέα σημεία και την σχέση

$$f[z_{2i}, z_{2i+1}] = f'(x_i) \quad (1.25)$$

που είναι η αντίστοιχη της $f[x_i, x_i] = f'(x_i)$.

Τέλος χρησιμοποιώντας τον τύπο παρεμβολής του Newton, το πολυώνυμο παρεμβολής Hermite μπορεί να παρασταθεί ως ακολούθως:

$$H_{2n+1}(x) = f(z_0) + \sum_{i=1}^{2n+1} f[z_0, z_1, \dots, z_i] \prod_{j=0}^{i-1} (x - z_j) \quad (1.26)$$

Παράδειγμα:

Να βρεθεί το πολυώνυμο παρεμβολής Hermite που παρεμβάλλει την συνάρτηση f στα σημεία $x_0 = -3, x_1 = 2$ και για το οποίο ισχύουν οι εξής συνθήκες παρεμβολής:

$$p(-3) = f(-3) = 2$$

$$p'(-3) = f'(-3) = 1$$

$$p(2) = f(2) = 3$$

$$p'(2) = f'(2) = 6$$

Λύση:

Ορίζουμε νέα ακολουθία σημείων με την βοήθεια της σχέσης

$$z_{2i} = z_{2i+1} = x_i, \text{ για } i = 0, 1$$

οπότε θα έχουμε:

$$\text{για } i = 0 \Rightarrow z_0 = -3 \text{ και } z_1 = -3$$

$$\text{για } i = 1 \Rightarrow z_2 = 2 \text{ και } z_3 = 2$$

Κατασκευάζουμε τον πίνακα διαιρεμένων διαφορών:

$z_0 = -3$	$f(z_0) = 2$			
		$f'(z_0) = 1$		
$z_1 = -3$	$f(z_0) = 2$		$f[z_0, z_1, z_2] = \frac{-1}{5}$	
		$f[z_1, z_2] = \frac{1}{5}$		$f[z_0, z_1, z_2, z_3] = \frac{33}{125}$
$z_2 = 2$	$f(z_2) = 3$		$f[z_1, z_2, z_3] = \frac{29}{25}$	
		$f'(z_2) = 6$		
$z_3 = 2$	$f(z_3) = 3$			

όπου $f'(z_0) = f[z_0, z_1]$ και $f'(z_2) = f[z_2, z_3]$.

Τότε από την σχέση (1.21) το πολυώνυμο Hermite προκύπτει να έχει την μορφή

$$p(z) = f(z_0) + f'(z_0)(z - z_0) + f[z_0, z_1, z_2](z - z_0)(z - z_1)$$

$$+ f[z_0, z_1, z_2, z_3](z - z_0)(z - z_1)(z - z_2)$$

ενώ αντικαθιστώντας τις τιμές παίρνουμε

$$p(z) = \frac{33}{125}z^3 + \frac{112}{125}z^2 - \frac{94}{125}z - \frac{149}{125}$$

που είναι και το επιθυμητό.

1.6 Σφάλμα παρεμβολής

Η προσέγγιση μιας συνάρτησης f από μια άλλη συνάρτηση g εισάγει κάποιο σφάλμα. Έτσι και στην πολυωνυμική παρεμβολή, είναι δυνατό η σχέση της παρεμβαλλόμενης συνάρτησης f και του πολυωνύμου παρεμβολής p να είναι

$$i, j = 1, 2 \dots n$$

οπότε έχουμε ένα **σφάλμα πολυωνυμικής παρεμβολής** ε τέτοιο ώστε

$$\varepsilon(x) = f(x) - p(x). \quad (1.25)$$

Το σφάλμα παρεμβολής συνδέεται άμεσα με την εκλογή του συνόλου των σημείων παρεμβολής.

Το σφάλμα παρεμβολής στην περίπτωση της παρεμβολής Newton δίνεται από τον τύπο

$$\varepsilon(x) = f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i) \quad (1.26)$$

για κάποιο $\xi \in (a, b)$, όπου $[a, b]$ το διάστημα στο οποίο ανήκουν τα διακεκριμένα σημεία x_0, x_1, \dots, x_n και $f \in C^{n+1}[a, b]$.

Αντίστοιχα για το σφάλμα του πολυωνύμου Hermite έχουμε:

$$\varepsilon(x) = f(x) - H_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{2n+1} (x - z_i) \quad (1.27)$$

για $\xi \in (a, b)$, $f \in C^{2n+2}[a, b]$, $z_{2i} = z_{2i+1} = x_i$ και $f[z_{2i}, z_{2i+1}] = f'(x_i)$,
 $i = 0, 1, \dots, n$

2. Ιδιοτιμές και Ιδιοδιανύσματα πίνακα

2.1. Ιδιοτιμές και Ιδιοδιανύσματα

Ορισμός 2.1. Έστω $m \in N$ και $A \in C^{m \times m}$ ένας τετραγωνικός πίνακας. Ένα μη-μηδενικό διάνυσμα $x \in C^m$ καλείται **Ιδιοδιανύσμα** του A και το $\lambda \in C$, **Ιδιοτιμή** που αντιστοιχεί στο διάνυσμα αυτό, αν:

$$Ax = \lambda x \quad (2.1)$$

Αυτό που φαίνεται από την (2.1) είναι ότι το γινόμενο Ax λειτουργεί ως βαθμωτό γινόμενο λx που εξαρτάται από τον πίνακα A και το διάνυσμα x . Ο υποχώρος S του C^m που αποτελείται από τα διανύσματα $x \in C^m$ τέτοια ώστε $Ax = \lambda x$ ονομάζεται **Ιδιοχώρος** του πίνακα A που αντιστοιχεί στην ιδιοτιμή λ και τα μη μηδενικά $x \in S$ είναι ιδιοδιανύσματα του A που αντιστοιχούν στην ιδιοτιμή λ .

Ορισμός 2.2. Το σύνολο όλων των ιδιοτιμών ενός πίνακα A καλείται **φάσμα** του A και είναι ένα υποσύνολο του C το οποίο συμβολίζεται ως $\Lambda(A)$.

Τα προβλήματα ιδιοτιμών έχουν ένα πολύ διαφορετικό χαρακτήρα από τα προβλήματα τετραγωνικών ή ορθογώνιων γραμμικών συστημάτων εξισώσεων. Για ένα σύστημα εξισώσεων, το πεδίο ορισμού μπορεί να είναι ένας χώρος και το σύνολο τιμών ένας άλλος.

Για παράδειγμα, έστω πίνακας A που αντιστοιχεί διανύσματα μήκους n με πολυωνυμικούς συντελεστές σε διανύσματα μήκους m απλών πολυωνύμων.

Το να αναζητήσουμε τις ιδιοτιμές ενός τέτοιου πίνακα A θα ήταν ανούσιο. Τα προβλήματα ιδιοτιμών έχουν νόημα μόνο όταν οι χώροι του συνόλου τιμών και το πεδίο ορισμού είναι οι ίδιοι. Αυτό αντανακλά το γεγονός ότι στις εφαρμογές, οι ιδιοτιμές χρησιμοποιούνται κυρίως όταν θέλουμε να συνθέσουμε ένα πίνακα αναδρομικά, είτε κατηγορηματικά ως δύναμη A^k είτε υπονοούμενα σε συναρτησιακή μορφή όπως: e^{tA} , όπως, π.χ., στην λύση διαφορικών εξισώσεων.

Γενικά μιλώντας, οι ιδιοτιμές και τα ιδιοδιανύσματα είναι χρήσιμα για δύο λόγους, έναν αλγορίθμικό και έναν που αφορά τη φυσική. Αλγορίθμικά, η ανάλυση ιδιοτιμών μπορεί να απλοποιήσει λύσεις σε ορισμένα προβλήματα, μετατρέποντας ένα περίπλοκο σύστημα σε συλλογή από βαθμωτά προβλήματα. Όσον αφορά τη φυσική, η ανάλυση ιδιοτιμών μπορεί να δώσει μια άποψη ως προς τη συμπεριφορά

συστημάτων που διέπονται από γραμμικές εξισώσεις, όπως, για παράδειγμα συστήματα δύο ή περισσότερων σωμάτων συνδεδεμένα με ελατήρια διαφορετικής τάσης το καθένα που αφορούν την κινηματική.

2.2. Διαγωνιοποίηση πινάκων

Διαγωνιοποίηση ενός τετραγωνικού πίνακα A καλείται η παραγοντοποίησή του, όταν φυσικά αυτή είναι δυνατή, σε

$$A = X \Lambda X^{-1} \quad (2.2)$$

όπου X είναι αντιστρέψιμος και Λ διαγώνιος πίνακας. Η παραπάνω σχέση, (2.2), μπορεί να γραφεί ισοδύναμα ως

$$AX = Q\Lambda \quad (2.3)$$

Δηλαδή

$$[A] \cdot [x_1|x_2| \cdots |x_m] = [x_1|x_2| \cdots |x_m] \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$$

Αυτό κάνει ξεκάθαρο ότι αν x_j είναι η j -στήλη του X και λ_j είναι το j -οστό στοιχείο του Λ , τότε, $Ax_j = \lambda_j x_j$. Έτσι η j στήλη του X είναι ένα ιδιοδιάνυσμα του A και το j -οστό διαγώνιο στοιχείο του Λ είναι η αντίστοιχη ιδιοτιμή.

Η διαγωνιοποίηση εκφράζει μια αλλαγή βάσης στις «συντεταγμένες του ιδιοδιανύσματος». Αν $Ax = b$ και $A = X \Lambda X^{-1}$ τότε έχουμε:

$$(X^{-1}b) = \Lambda(X^{-1}x) \quad (2.4)$$

Έτσι για να υπολογίσουμε το Ax μπορούμε να επεκτείνουμε το x με βάση τις στήλες του X , να εφαρμόσουμε τον Λ και να ερμηνεύσουμε το αποτέλεσμα σαν ένα διάνυσμα συντελεστών ενός γραμμικού συνδυασμού των στηλών του X .

2.3. Γεωμετρική Πολλαπλότητα

Όπως ήδη αναφέρθηκε, το σύνολο των ιδιοδιανυσμάτων που αντιστοιχούν σε μια ιδιοτιμή, μαζί με το μηδενικό διάνυσμα, αποτελούν έναν υπόχωρο του C^m γνωστό ως ιδιόχωρο. Έστω μια ιδιοτιμή ενός πίνακα A , τότε συμβολίζουμε τον αντίστοιχο ιδιόχωρο με ε_λ . Ο ιδιόχωρος ε_λ αποτελεί ένα παράδειγμα ενός αναλλοίωτου υποχώρου του A , δηλαδή $A\varepsilon_\lambda \subset \varepsilon_\lambda$ (καθώς, αν $x \in A\varepsilon_\lambda$, τότε $x = Ay = \lambda y$ για κάποιο ιδιοδιανυσμα y . Τότε όμως $Ax = A\lambda y = \lambda Ay = \lambda x$. Άρα τελικά $x \in \varepsilon_\lambda$)

Η διάσταση του ε_λ μπορεί να ερμηνευτεί ως ο μέγιστος αριθμός γραμμικώς ανεξαρτήτων ιδιοδιανυσμάτων που αντιστοιχούν στην ίδια ιδιοτιμή λ που μπορεί να θρεθεί. Αυτός ο αριθμός είναι γνωστός ως γεωμετρική πολλαπλότητα του λ . Η γεωμετρική πολλαπλότητα μπορεί να περιγραφεί επίσης σαν τη διάσταση του μηδενόχωρου του $A - \lambda I$ από τη στιγμή που αυτός ο μηδενόχωρος είναι ο ίδιος ο ε_λ .

2.4. Χαρακτηριστικό Πολυώνυμο

Το χαρακτηριστικό πολυώνυμο του $A \in C^{m \times m}$ που συμβολίζεται ως p_A ή απλούστερα p , είναι το πολυώνυμο βαθμού m που ορίζεται ως :

$$p_A(z) = \det(zI - A) \quad (2.5)$$

Λόγω της συγκεκριμένης θέσης του προσήμου $(-)$, το p είναι μονικό (δηλαδή, ο συντελεστής του μεγιστοβάθμιου όρου, βαθμού m , είναι 1).

Θεωρημα 2.4. Το λ είναι μια ιδιοτιμή του πίνακα A αν και μόνο αν $p_A(\lambda) = 0$.

Απόδειξη: Το ζητούμενο προκύπτει από τον ορισμό της ιδιοτιμής : λ είναι ιδιοτιμή αν και μόνο αν υπάρχει μη-μηδενικό διάνυσμα τέτοιο ώστε $\lambda x - Ax = 0$. Αυτό όμως είναι ισοδύναμο με το ότι ο πίνακας $\lambda I - A$ είναι μη αντιστρέψιμος και αυτό ισχύει αν και μόνο αν $\det(\lambda I - A) = 0$.

Το Θεώρημα 2.4 έχει μια σημαντική συνέπεια. Ακόμα και αν ένας πίνακας είναι πραγματικός, κάποιες από τις ιδιοτιμές του μπορεί να είναι μιγαδικές. (Για παράδειγμα, ο πίνακας:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

είναι πραγματικός αλλά οι ρίζες του χαρακτηριστικού του πολυωνύμου, δηλαδή οι ιδιοτιμές του είναι οι $\pm i$ που είναι μιγαδικές.) Στη φυσική, αυτό σχετίζεται με το φαινόμενο στο οποίο, πραγματικά δυναμικά συστήματα μπορεί να παρουσιάζουν παλινδρομικές κινήσεις όπως και αυξήσεις και μειώσεις. Αλγορίθμικά, σημαίνει ότι ακόμα και αν τα δεδομένα εισόδου σε ένα πίνακα προβλήματος ιδιοτιμών είναι πραγματικά, τα αποτελέσματα, (ιδιοτιμές και ιδιοδιανύσματα), μπορεί να είναι μιγαδικά.

1.5. Αλγεβρική Πολλαπλότητα

Από το θεμελιώδες θεώρημα της Άλγεβρας, μπορούμε να γράψουμε το p_A στη μορφή:

$$p_A(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m) \quad (2.6)$$

για κάποια λ_j . Σύμφωνα με το Θεώρημα 2.4, κάθε λ_j είναι ιδιοτιμή του A και όλες οι ιδιοτιμές του A είναι κάποιο $\lambda_j \in C$. Γενικά, μια ιδιοτιμή μπορεί να εμφανιστεί περισσότερες από μία φορές. Ορίζουμε ως **αλγεβρική πολλαπλότητα** μιας ιδιοτιμής λ του A , την πολλαπλότητα με την οποία εμφανίζεται ως ρίζα του πολυωνύμου p_A . Μια ιδιοτιμή είναι απλή αν η αλγεβρική της πολλαπλότητα είναι 1.

Το χαρακτηριστικό πολυώνυμο μας δίδει έναν εύκολο τρόπο υπολογισμού του πλήθους των ιδιοτιμών ενός πίνακα.

Σημείωση: Γενικά κάθε πίνακας έχει τουλάχιστον μία ιδιοτιμή. (Η ιδιοτιμή είναι μοναδική όταν η πολλαπλότητά της είναι όση και ο βαθμός του χαρακτηριστικού πολυωνύμου του πίνακα δηλαδή όση και η διάσταση του πίνακα.)

Παράδειγμα 1°:

Να βρεθούν οι ιδιοτιμές και τα ιδιοδιανύσματα του $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Σύμφωνα με τη θεωρία το χαρακτηριστικό πολυώνυμο του πίνακα είναι:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda) \begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = ((3-\lambda)^2 - (-2) \cdot (-2))(5-\lambda) =$$

$$= (\lambda^2 - 6\lambda + 9 - 4)(5-\lambda) = -(\lambda-5)^2(\lambda-1)$$

Οπότε έχουμε δύο ιδιοτιμές την $\lambda=1$ και τη διπλής αλγεβρικής πολλαπλότητας $\lambda=5$.

Για $\lambda=1$ τα ιδιοδιανύσματα προκύπτουν από την λύση του συστήματος:

$$(A - 1 \cdot I) \cdot \tilde{x} = 0 \Leftrightarrow \begin{bmatrix} 3-1 & -2 & 0 \\ -2 & 3-1 & 0 \\ 0 & 0 & 5-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{l} 2x_1 - 2x_2 = 0 \\ -2x_1 + 2x_2 = 0 \\ 4x_3 = 0 \end{array} \Leftrightarrow \begin{array}{l} x_1 = x_2 \\ x_3 = 0 \end{array}$$

Οπότε το ιδιοδιάνυσμα που αντιστοιχεί σε αυτήν την ιδιοτιμή είναι της μορφής:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Για $\lambda=5$ τα ιδιοδιανύσματα προκύπτουν από την λύση του συστήματος:

$$(\mathbf{A} - 5 \cdot I) \cdot \tilde{x} = 0 \Leftrightarrow \begin{bmatrix} 3-5 & -2 & 0 \\ -2 & 3-5 & 0 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{l} -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_2 = 0 \\ 0x_3 = 0 \end{array} \Leftrightarrow \begin{array}{l} x_1 = -x_2 \\ x_3 \text{ ανθαίρετο} \end{array}$$

Οπότε τα ιδιοδιανύσματα που αντιστοιχούν σε αυτήν την ιδιοτιμή είναι της μορφής:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

και τα αντίστοιχα ιδιοδιανύσματα της ιδιοτιμής είναι τα

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Παράδειγμα 2º:

Να βρεθούν οι ιδιοτιμές και τα ιδιοδιανύσματα του πίνακα $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

Το χαρακτηριστικό πολυώνυμο έχει ρίζες τις ιδιοτιμές του \mathbf{A} , $\lambda_1 = i, \lambda_2 = -i$.

Θα βρούμε τώρα τα ιδιοδιανύσματα.

$$\text{Για } \lambda_1 = i \text{ έχουμε: } (A - \lambda_1 I) \cdot \tilde{x} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Το σύστημα έχει λύση } x_1 = ix_2 \text{ οπότε } \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\text{και οπότε το αντίστοιχο ιδιοδιάνυσμα είναι: } v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

$$\text{Για } \lambda_2 = -i \text{ έχουμε: } (A - \lambda_2 I) \cdot \tilde{x} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Το σύστημα έχει λύση } x_1 = -ix_2 \text{ οπότε } \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\text{και οπότε το αντίστοιχο ιδιοδιάνυσμα είναι: } v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

3. Ρίζες πολυωνύμου, βαθμός πολλαπλότητας με τη χρήση παραγώγων

Μια συνάρτηση λέγεται πολυωνυμική αν υπάρχουν πραγματικοί αριθμοί $\alpha_0, \alpha_1, \dots, \alpha_v$, τέτοιοι ώστε :

$$f(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_v x^v \text{ για κάθε } x \in R$$

Ο ελάχιστος εκθέτης ν για τον οποίο συμβαίνει αυτό ονομάζεται **βαθμός της f** και συμβολίζεται $\deg f$.

Είναι φανερό ότι η μηδενική συνάρτηση (που είναι πολυωνυμική) δεν έχει βαθμό.

Οι αριθμοί $\alpha_0, \alpha_1, \dots, \alpha_v$ στην παραπάνω γραφή ονομάζονται **συντελεστές** της f και προσδιορίζουν κατά μοναδικό τρόπο την f.

Ο τύπος Taylor για πολυωνυμικές συναρτήσεις:

Θεωρούμε μια πολυωνυμική συνάρτηση $f \in R_v[x]$ και έναν πραγματικό αριθμό ρ .

Αφού οι

$$1, (x - p), (x - p)^2, \dots, (x - p)^v$$

αποτελούν βάση του $R_v[x]$ η f θα γράφεται κατά μοναδικό τρόπο ως γραμικός συνδυασμός τους:

$$f(x) = \lambda_0 + \lambda_1(x - \rho) + \dots + \lambda_v(x - \rho)^v$$

Ας υπολογίσουμε τους συντελεστές $\lambda_0, \lambda_1, \dots, \lambda_v$. Για τον σκοπό αυτό παραγωγίζουμε κ-φορές την παραπάνω σχέση οπότε βρίσκουμε

$$f^\kappa(x) = \kappa! \lambda_\kappa + (x - \rho)g(x) , \quad g \in R_v[x] \quad \kappa=0,1,\dots,v$$

Βάζοντας $x=\rho$ παίρνουμε

$$\lambda^\kappa = \frac{f^\kappa(\rho)}{\kappa!} \quad \kappa=0,1,\dots,v$$

Έτσι τελικά

$$f(x) = f(p) + \frac{f'(p)}{1!}(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \dots + \frac{f^{(v)}(p)}{v!}(x-p)^v$$

που είναι ο τύπος του Taylor για την f , στο σημείο $\rho \in R$.

Λέμε λοιπόν ότι ο πραγματικός αριθμός ρ είναι ρίζα της πολυωνυμικής συνάρτησης f αν συμβαίνει $f(p)=0$.

Ορισμός 3.1: Το ρ είναι **ρίζα** της f τότε και μόνο όταν $f(x)=(x-p)\cdot g(x)$, $g \in R[x]$.

Απόδειξη:

Από τον τύπο του Taylor αν λάβουμε υπόψη ότι $f(p)=0$ και από τους υπόλοιπους προσθετέους βγάλουμε κοινό παράγοντα το $(x-\rho)$ βρίσκουμε

$$f(x)=(x-p)\cdot g(x)$$

όπως το θέλαμε. Το αντίστροφο είναι άμεσο. Αν η προηγούμενη σχέση ισχύει, τότε βάζοντας $x=p$ παίρνουμε

$$f(p)=(p-p)\cdot g(p)=0$$

δηλαδή το ρ είναι ρίζα της $f(x)$.

Ορισμός 3.2: Μία πολυωνυμική συνάρτηση f βαθμού v δεν μπορεί να έχει περισσότερες από v ανά δύο διαφορετικές ρίζες.

Απόδειξη:

Πράγματι, ας είναι

$$\rho_1 < \rho < \dots < \rho_{v+1}$$

$n+1$ ρίζες της f . Σύμφωνα με την προηγούμενη πρόταση

$$f(x) = (x - p_1) \cdot g(x)$$

βάζοντας στη σχέση αυτή $x = p_2$ παίρνουμε

$$0 = f(p_2) = (p_2 - p_1) \cdot g(p_2)$$

απ' όπου $g(p_2) = 0$ (γιατί $p_2 - p_1 \neq 0$). Θα έχουμε λοιπόν

$$g(x) = (x - p_2) \cdot h(x)$$

δηλαδή

$$f(x) = (x - p_1)(x - p_2) \cdot h(x)$$

Συνεχίζοντας μ' αυτόν τον τρόπο καταλήγουμε στη γραφή

$$f(x) = (x - p_1)(x - p_2) \dots (x - p_{n+1}) \cdot \omega(x).$$

Το αριστερό μέλος στην παραπάνω ισότητα είναι μία πολυωνυμική συνάρτηση βαθμού n (από την υπόθεση) ενώ το δεξιό μέλος έχει βαθμό τουλάχιστον $n+1$, αντίφαση.

Στην αντίφαση αυτή πέσαμε γιατί ακριβώς υποθέσαμε ότι η $f(x)$ έχει περισσότερες από n διακεκριμένες ρίζες.

Το προηγούμενο θεώρημα έχει την εξής σημαντική συνέπεια:

Αν υπάρχει ακολουθία αριθμών

$$\rho_1 < \rho_2 < \dots < \rho_n < \dots$$

που μηδενίζουν την πολυωνυμική συνάρτηση f , τότε $f=0$.

Πράγματι, διότι αν ήταν $f \neq 0$, τότε η f θα είχε κάποιο βαθμό έστω n , και περισσότερες από n ρίζες, πράγμα αδύνατο.

Λέμε ότι ο πραγματικός αριθμός ρ είναι ρίζα βαθμού πολλαπλότητας κ της πολυωνυμικής συνάρτησης f αν συμβαίνει:

$$f(x) = (x - p)^\kappa \cdot g(x) \text{ και } g(p) \neq 0.$$

Θεώρημα 3.3: Το ρ είναι ρίζα πολλαπλότητας κ της f τότε και μόνο όταν

$$f(p) = 0, f'(p) = 0, \dots, f^{(\kappa-1)}(p) = 0 \quad \text{ενώ} \quad f^\kappa(p) \neq 0.$$

Απόδειξη:

Αν οι παραπάνω συνθήκες ικανοποιούνται ο βαθμός της f είναι $\geq \kappa$ και ο τύπος του Taylor δίνει

$$\begin{aligned} f(x) &= \frac{f^\kappa(\rho)}{\kappa!} (x - \rho)^\kappa + \frac{f^{\kappa+1}(\rho)}{(\kappa-1)!} (x - \rho)^{\kappa+1} + \dots + \frac{f^{(\nu)}(\rho)}{\nu!} (x - \rho)^\nu \\ &= (x - \rho)^\kappa \cdot g(x) \end{aligned}$$

όπου

$$g(p) = \frac{f^\kappa(\rho)}{\kappa!} \neq 0.$$

Σύμφωνα με τον ορισμό λοιπόν το ρ είναι ρίζα βαθμού πολλαπλότητας κ της f . Το αντίστροφο τώρα: αν συμβαίνει

$$f(x) = (x - p)^\kappa \cdot g(x) \text{ με } g(p) \neq 0$$

τότε παραγωγίζοντας την f παίρνουμε

$$\begin{aligned} f'(x) &= \kappa(x - p)^{\kappa-1} \cdot g(x) + (x - p)^\kappa \cdot g'(x) \\ &= (x - p)[\kappa \cdot g(x) + (x - p) \cdot g'(x)] \\ &= (x - p)^{\kappa-1} \cdot h(x) \end{aligned}$$

όπου βάλαμε

$$h(x) = \kappa \cdot g(x) + (x - p) \cdot g'(x).$$

Είναι

$$h(p) = \kappa \cdot g(p) \neq 0.$$

Με άλλα λόγια έχουμε

$$f'(x) = (x - p)^{\kappa-1} \cdot h(x) \quad \text{και} \quad h(p) \neq 0$$

δηλαδή το ρ είναι ρίζα βαθμού πολλαπλότητας $\kappa-1$ της $f'(x)$.

Συνεχίζοντας με τον τρόπο αυτό βρίσκουμε ότι το ρ είναι ρίζα πολλαπλότητας $\kappa-2$ της $f''(x)$, κ.ο.κ. το ρ είναι ρίζα πολλαπλότητας 1 της $f^{\kappa-1}(x)$ ενώ $f^\kappa(\rho) \neq 0$ όπως το θέλαμε.

Βαθμό πολλαπλότητας, γενικεύοντας τα παραπάνω, ορίζουμε το βαθμό πολλαπλότητας μιας ρίζας για μια τυχαία συνάρτηση.

Θα λέμε ότι μια τυχαία συνάρτηση έχει ρίζα x_0 , βαθμού πολλαπλότητας k αν και μόνο αν :

$$f(x_0) = 0$$

$$f'(x_0) = 0$$

\vdots

$$f^{(k-1)}(x_0) = 0$$

$$\text{και} \quad f^{(k)}(x_0) \neq 0$$

4. «Τριγωνομετρία Πινάκων»

4.1 Αριθμητική παρεμβολή

ΟΡΙΣΜΟΣ

Αν $A = [a]$ είναι ένας 1×1 πίνακας τότε ορίζουμε:

$$\text{Sin}[A] = [\text{Sin}[a]]$$

$$\text{Cos}[A] = [\text{Cos}[a]]$$

$$\text{Tan}[A] = [\text{Tan}[a]]$$

$$\text{Cot}[A] = [\text{Cot}[a]]$$

με την προϋπόθεση ότι ορίζονται οι τριγωνομετρικοί αριθμοί της εφαπτομένης και συνεφαπτομένης.

Αν ο αριθμός a είναι μιγαδικός με $a = x + yi$ και $x, y \in R$ τότε οι τριγωνομετρικοί αριθμοί του a ορίζονται από τις σχέσεις:

$$\text{Sin}[a] = \text{Cosh}[y]\text{Sin}[x] + \text{Cos}[x]\text{Sinh}[y]i$$

$$\text{Cos}[a] = \text{Cos}[x]\text{Cosh}[y] - \text{Sin}[x]\text{Sinh}[y]i$$

$$\text{Tan}[a] = \frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}i$$

$$\text{Cot}[a] = \frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{\text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}i$$

όπου οι υπερβολικοί τριγωνομετρικοί αριθμοί ορίζονται ως εξής:

$$\text{Sinh}[x] = \frac{e^x - e^{-x}}{2}$$

$$Cosh[x] = \frac{e^x + e^{-x}}{2}$$

$$Tanh[x] = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$Coth[x] = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

4.1.1. Ορισμός πίνακα 2x2

Έστω $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ένας 2x2 πίνακας. Ονομάζεται χαρακτηριστικό πολυώνυμο του πίνακα αυτού το πολυώνυμο:

$$cp(x) = \begin{vmatrix} \alpha - x & \beta \\ \gamma & \delta - x \end{vmatrix} = x^2 - (\alpha + \delta)x + (\alpha\delta - \beta\gamma)$$

Συμβολίζω με λ_1, λ_2 τις ρίζες του πολυωνύμου αυτού, οι οποίες λέγονται ιδιοτιμές του πίνακα A.

Ορισμός του πίνακα $Sin[A]$

Διακρίνω τις περιπτώσεις:

1^η περίπτωση:

Ο πίνακας A έχει **δύο άνισες ιδιοτιμές** λ_1, λ_2 .

Δεδομένα αριθμητικής παρεμβολής:

$$(\lambda_1, Sin[\lambda_1]), (\lambda_2, Sin[\lambda_2])$$

Πολυώνυμο παρεμβολής:

$$p(x) = \frac{Sin[\lambda_2] - Sin[\lambda_1]}{\lambda_2 - \lambda_1} x + \frac{\lambda_2 Sin[\lambda_1] - \lambda_1 Sin[\lambda_2]}{\lambda_2 - \lambda_1}$$

Ορίζω ως $\text{Sin}[A] = p(A)$ δηλαδή :

$$\text{Sin}[A] = \frac{\text{Sin}[\lambda_2] - \text{Sin}[\lambda_1]}{\lambda_2 - \lambda_1} A + \frac{\lambda_2 \text{Sin}[\lambda_1] - \lambda_1 \text{Sin}[\lambda_2]}{\lambda_2 - \lambda_1} I$$

και μετά τις πράξεις έχουμε :

$$\text{Sin}[A] = \begin{bmatrix} \frac{(a - \lambda_1) \text{Sin}[\lambda_2] - (a - \lambda_2) \text{Sin}[\lambda_1]}{\lambda_2 - \lambda_1} & \frac{\beta(\text{Sin}[\lambda_2] - \text{Sin}[\lambda_1])}{\lambda_2 - \lambda_1} \\ \frac{\gamma(\text{Sin}[\lambda_2] - \text{Sin}[\lambda_1])}{\lambda_2 - \lambda_1} & \frac{(\delta - \lambda_1) \text{Sin}[\lambda_2] - (\delta - \lambda_2) \text{Sin}[\lambda_1]}{\lambda_2 - \lambda_1} \end{bmatrix}$$

2^η περίπτωση:

Ο πίνακας Α έχει **δύο ίσες ιδιοτιμές** λ_1, λ_2 .

Δεδομένα αριθμητικής παρεμβολής :

$$(\lambda_1, \{\text{Sin}[\lambda_1], \text{Cos}[\lambda_1]\})$$

δηλαδή αναζητώ το πρωτοβάθμιο πολυώνυμο με τις ιδιότητες:

$$p(\lambda_1) = \text{Sin}[\lambda_1] \text{ και } p'(\lambda_1) = \text{Cos}[\lambda_1]$$

Πολυώνυμο παρεμβολής :

$$p(x) = \text{Cos}[\lambda_1]x + \text{Sin}[\lambda_1] - \lambda_1 \text{Cos}[\lambda_1].$$

Ορίζω ως $\text{Sin}[A] = p(A)$ δηλαδή

$$\text{Sin}[A] = \text{Cos}[\lambda_1]A + (\text{Sin}[\lambda_1] - \lambda_1 \text{Cos}[\lambda_1])I$$

και μετά τις πράξεις έχουμε :

$$Sin[A] = \begin{bmatrix} a\cos[\lambda_1] + \sin[\lambda_1] - \lambda_1 \cos[\lambda_1] & \beta \cos[\lambda_1] \\ \gamma \cos[\lambda_1] & \delta \cos[\lambda_1] + \sin[\lambda_1] - \lambda_1 \cos[\lambda_1] \end{bmatrix}$$

Όμοια αν ο πίνακας A έχει δύο άνισες ιδιοτιμές είναι:

$$Cos[A] = \begin{bmatrix} \frac{(a - \lambda_1)\cos[\lambda_2] - (a - \lambda_2)\cos[\lambda_1]}{\lambda_2 - \lambda_1} & \frac{\beta(\cos[\lambda_2] - \cos[\lambda_1])}{\lambda_2 - \lambda_1} \\ \frac{\gamma(\cos[\lambda_2] - \cos[\lambda_1])}{\lambda_2 - \lambda_1} & \frac{(\delta - \lambda_1)\cos[\lambda_2] - (\delta - \lambda_2)\cos[\lambda_1]}{\lambda_2 - \lambda_1} \end{bmatrix}$$

και αν έχει δύο ίσες ιδιοτιμές είναι :

$$Cos[A] = \begin{bmatrix} -a\sin[\lambda_1] + \cos[\lambda_1] + \lambda_1 \sin[\lambda_1] & -\beta \sin[\lambda_1] \\ -\gamma \sin[\lambda_1] & -\delta \sin[\lambda_1] + \cos[\lambda_1] + \lambda_1 \sin[\lambda_1] \end{bmatrix}$$

Όμοια αν ο πίνακας έχει δύο άνισες ιδιοτιμές είναι:

$$Tan[A] = \begin{bmatrix} \frac{(a - \lambda_1)\tan[\lambda_2] - (a - \lambda_2)\tan[\lambda_1]}{\lambda_2 - \lambda_1} & \frac{\beta(\tan[\lambda_2] - \tan[\lambda_1])}{\lambda_2 - \lambda_1} \\ \frac{\gamma(\tan[\lambda_2] - \tan[\lambda_1])}{\lambda_2 - \lambda_1} & \frac{(\delta - \lambda_1)\tan[\lambda_2] - (\delta - \lambda_2)\tan[\lambda_1]}{\lambda_2 - \lambda_1} \end{bmatrix}$$

και αν έχει δύο ίσες ιδιοτιμές είναι :

$$Tan[A] = \begin{bmatrix} a(1 + \tan^2[\lambda_1]) + \tan[\lambda_1] - \lambda_1(1 + \tan^2[\lambda_1]) & \beta(1 + \tan^2[\lambda_1]) \\ \gamma(1 + \tan^2[\lambda_1]) & \delta(1 + \tan^2[\lambda_1]) + \tan[\lambda_1] - \lambda_1(1 + \tan^2[\lambda_1]) \end{bmatrix}$$

Τέλος αν ο πίνακας A έχει δύο άνισες ιδιοτιμές είναι :

$$Cot[A] = \begin{bmatrix} \frac{(a - \lambda_1)Cot[\lambda_2] - (a - \lambda_2)Cot[\lambda_1]}{\lambda_2 - \lambda_1} & \frac{\beta(Cot[\lambda_2] - Cot[\lambda_1])}{\lambda_2 - \lambda_1} \\ \frac{\gamma(Cot[\lambda_2] - Cot[\lambda_1])}{\lambda_2 - \lambda_1} & \frac{(\delta - \lambda_1)Cot[\lambda_2] - (\delta - \lambda_2)Cot[\lambda_1]}{\lambda_2 - \lambda_1} \end{bmatrix}$$

Ενώ αν έχει δύο ίσες ιδιοτιμές είναι:

$$Cot[A] = \begin{bmatrix} a(-1 - Cot^2[\lambda_1] + Cot[\lambda_1] - \lambda_1(1 + Cot^2[\lambda_1])) & \beta(-1 - Cot^2[\lambda_1]) \\ \gamma(-1 - Cot^2[\lambda_1]) & \delta(-1 - Cot^2[\lambda_1]) + Cot[\lambda_1] + \lambda_1(1 + Cot^2[\lambda_1]) \end{bmatrix}$$

4.1.2. Ορισμός πίνακα 3×3

Έστω $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ένας 3×3 πίνακας. Ονομάζεται χαρακτηριστικό

πολυώνυμο του πίνακα αυτού το πολυώνυμο:

$$p(x) = Det(A - xI_3)$$

Συμβολίζω με $\lambda_1, \lambda_2, \lambda_3$ τις ρίζες του πολυωνύμου αυτού, οι οποίες λέγονται ιδιοτιμές του πίνακα A.

Ορισμός του $Sin[A]$

Διακρίνω τις περιπτώσεις :

1^η περίπτωση:

Ο πίνακας Α έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ διαφορετικές ανά δυο .

Δεδομένα αριθμητικής παρεμβολής

$$(\lambda_1, \sin[\lambda_1]), (\lambda_2, \sin[\lambda_2]), (\lambda_3, \sin[\lambda_3])$$

Πολυώνυμο παρεμβολής

Χρησιμοποιώντας την μέθοδο Newton ή μέθοδο Διαιρεμένων Διαφορών (1.4)

βρίσκω το πολυώνυμο παρεμβολής $p(x)$:

Newton

X _i	F[X _i]	F[X _i , X _{i+1}]	F[X _i , X _{i+1} , X _{i+2}]
X ₀ =λ ₁	Sin[λ ₁]		
		F[X ₀ , X ₁]	
X ₁ =λ ₂	Sin[λ ₂]		F[X ₀ , X ₁ , X ₂]
		F[X ₁ , X ₂]	
X ₂ =λ ₃	Sin[λ ₃]		

$$F[x_0, x_1] = \frac{F[x_1] - F[x_0]}{x_1 - x_0} = \frac{\sin[\lambda_2] - \sin[\lambda_1]}{\lambda_2 - \lambda_1}$$

$$F[x_1, x_2] = \frac{F[x_2] - F[x_1]}{x_2 - x_1} = \frac{\sin[\lambda_3] - \sin[\lambda_2]}{\lambda_3 - \lambda_2}$$

$$F[x_0, x_1, x_2] = \frac{F[x_1, x_2] - F[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\sin[\lambda_3] - \sin[\lambda_2]}{\lambda_3 - \lambda_2} - \frac{\sin[\lambda_2] - \sin[\lambda_1]}{\lambda_2 - \lambda_1}}{\lambda_3 - \lambda_1} =$$

$$\begin{aligned} p_{11} = & \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ & \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ & \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} \end{aligned}$$

Το πολυώνυμο παρεμβολής προκύπτει από τον τύπο:

$$\begin{aligned} p(x) = & F[x_0] + F[x_0, x_1](x - x_0) + F[x_0, x_1, x_2](x - x_0)(x - x_1) = \\ = & \sin[\lambda_1] + \frac{\sin[\lambda_2] - \sin[\lambda_1]}{(\lambda_2 - \lambda_1)}(x - \lambda_1) + \\ + & \frac{(\lambda_2 - \lambda_1)(\sin[\lambda_3] - \sin[\lambda_2]) - (\lambda_3 - \lambda_2)(\sin[\lambda_2] - \sin[\lambda_1])}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}(x - \lambda_1)(x - \lambda_2) \end{aligned}$$

Μετά από πράξεις έχουμε:

$$\begin{aligned} p(x) = & \frac{(\lambda_3 - \lambda_2)\sin[\lambda_1] + (\lambda_1 - \lambda_3)\sin[\lambda_2] + (\lambda_2 - \lambda_1)\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}x^2 + \\ + & \frac{(\lambda_2^2 - \lambda_3^2)\sin[\lambda_1] + (\lambda_3^2 - \lambda_1^2)\sin[\lambda_2] + (\lambda_1^2 - \lambda_2^2)\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}x + \\ + & \frac{\lambda_2\lambda_3(\lambda_3 - \lambda_2)\sin[\lambda_1] + \lambda_1\lambda_3(\lambda_1 - \lambda_3)\sin[\lambda_2] + \lambda_1\lambda_2(\lambda_2 - \lambda_1)\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} \end{aligned}$$

Ορίζω ως $\text{Sin}[A] = p(A)$ δηλαδή :

$$\begin{aligned}\text{Sin}[A] &= \frac{(\lambda_3 - \lambda_2) \text{Sin}[\lambda_1] + (\lambda_1 - \lambda_3) \text{Sin}[\lambda_2] + (\lambda_2 - \lambda_1) \text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} A^2 + \\ &+ \frac{(\lambda_2^2 - \lambda_3^2) \text{Sin}[\lambda_1] + (\lambda_3^2 - \lambda_1^2) \text{Sin}[\lambda_2] + (\lambda_1^2 - \lambda_2^2) \text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} A + \\ &+ \frac{(\lambda_3 - \lambda_2) \lambda_2 \lambda_3 \text{Sin}[\lambda_1] + \lambda_1 \lambda_3 (\lambda_1 - \lambda_3) \text{Sin}[\lambda_2] + \lambda_1 \lambda_2 (\lambda_2 - \lambda_1) \text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} I_3\end{aligned}$$

και μετά από πράξεις έχουμε:

$$\text{Sin}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2 \lambda_3 (\lambda_3 - \lambda_2)] \text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1 \lambda_3 (\lambda_1 - \lambda_3)] \text{Sin}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1 \lambda_2 (\lambda_2 - \lambda_1)] \text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{21} = \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)] \text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)] \text{Sin}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{31} = \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{12} = \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{13} = \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{22} = \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{23} = \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{32} = \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{33} = \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

Όμοια αν ο πίνακας A έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ διαφορετικές ανά δυο είναι:

$$\cos[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2)] + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) - a_{11}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{21} = \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) - a_{21}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{31} = \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{21} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{21} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{21} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{21}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{21}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{21}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) + a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$p_{32} = \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{33} = \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

Όμοια αν ο πίνακας A έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ διαφορετικές ανά δυο είναι:

$$\text{Tan}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) - a_{11}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$\begin{aligned}
p_{21} = & \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) - a_{21}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$p_{22} = \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) + a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{23} = \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{32} = \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{33} = \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

Όμοια αν ο πίνακας A έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ διαφορετικές ανά δυο είναι:

$$\text{Cot}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) - a_{11}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{21} = \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) - a_{21}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{31} = \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} +$$

$$\frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) + a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}
\end{aligned}$$

$$p_{32} = \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

$$p_{33} = \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}$$

2^η περίπτωση:

Ο πίνακας Α έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 \neq \lambda_3$.

Δεδομένα αριθμητικής παρεμβολής

$$\{\{\lambda_1, \{Sin[\lambda_1], Cos[\lambda_1]\}\}, \{\lambda_3, Sin[\lambda_3]\}\}$$

Πολυώνυμο παρεμβολής με τις ιδιότητες : $p(\lambda_1) = Sin[\lambda_1]$, $p'(\lambda_1) = Cos[\lambda_1]$.

Χρησιμοποιώντας την μέθοδο Hermite βρίσκω το πολυώνυμο παρεμβολής $p(x)$:

Hermite

X_i	$F[X_i]$	$F[X_i, X_{i+1}]$	$F[X_i, X_{i+1}, X_{i+2}]$
$X_0 = \lambda_1$	$\text{Sin}[\lambda_1]$		
		$F[X_0, X_1]$	
$X_1 = \lambda_1$	$\text{Sin}[\lambda_1]$		$F[X_0, X_1, X_2]$
		$F[X_1, X_2]$	
$X_2 = \lambda_3$	$\text{Sin}[\lambda_3]$		

όπου:

$$F[x_0, x_1] = \frac{F[x_1] - F[x_0]}{x_1 - x_0} = p'(x_0) = p'(\lambda_1) = \text{Cos}[\lambda_1]$$

$$F[x_1, x_2] = \frac{F[x_2] - F[x_1]}{x_2 - x_1} = \frac{\text{Sin}[\lambda_3] - \text{Sin}[\lambda_1]}{\lambda_3 - \lambda_1}$$

$$\begin{aligned} F[x_0, x_1, x_3] &= \frac{F[x_1, x_2] - F[x_0, x_1]}{x_2 - x_1} = \frac{\text{Sin}[\lambda_3] - \text{Sin}[\lambda_1]}{\lambda_3 - \lambda_1} - \text{Cos}[\lambda_1] = \\ &= \frac{\text{Sin}[\lambda_3] - \text{Sin}[\lambda_1] - (\lambda_3 - \lambda_1)\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} \end{aligned}$$

Το πολυώνυμο παρεμβολής προκύπτει από τον τύπο:

$$p(x) = F[x_0] + F[x_0, x_1](x - x_0) + F[x_0, x_1, x_2](x - x_0)(x - x_1) =$$

$$\text{Sin}[\lambda_1] + \text{Cos}[\lambda_1](x - \lambda_1) + \frac{\text{Sin}[\lambda_3] - \text{Sin}[\lambda_1] - (\lambda_3 - \lambda_1)\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)^2}(x - \lambda_1)^2$$

Μετά από πράξεις έχουμε:

$$\begin{aligned}
p(x) = & \frac{-\sin[\lambda_1] + \sin[\lambda_3] - (\lambda_3 - \lambda_1)\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} x^2 + \\
& \frac{2\lambda_1 \sin[\lambda_1] - 2\lambda_1 \sin[\lambda_3] + (\lambda_3^2 - \lambda_1^2)\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} x + \\
& \frac{\lambda_3(\lambda_3 - 2\lambda_1)\sin[\lambda_1] + \lambda_1^2 \sin[\lambda_3] - \lambda_1 \lambda_3(\lambda_3 - \lambda_1)\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2}
\end{aligned}$$

Ορίζω $\sin[A] = p(A)$ δηλαδή:

$$\begin{aligned}
\sin[A] = & \frac{-\sin[\lambda_1] + \sin[\lambda_3] - (\lambda_3 - \lambda_1)\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} A^2 + \\
& \frac{2\lambda_1 \sin[\lambda_1] - 2\lambda_1 \sin[\lambda_3] + (\lambda_3^2 - \lambda_1^2)\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} A + \\
& \frac{\lambda_3(\lambda_3 - 2\lambda_1)\sin[\lambda_1] + \lambda_1^2 \sin[\lambda_3] - \lambda_1 \lambda_3(\lambda_3 - \lambda_1)\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} I_3
\end{aligned}$$

και μετά από πράξεις έχουμε:

$$\sin[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$\begin{aligned}
p_{11} = & \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)}
\end{aligned}$$

$$p_{21} = \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{31} = \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{12} = \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{22} = \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{32} = \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{13} = \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) - 2\lambda_1\alpha_{13}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{23} = \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{23}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{23}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{33} = \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{33} + \lambda_3(\lambda_3 - 2\lambda_1)] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{33} + \lambda_1^2] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + \alpha_{33}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

Όμοια αν ο πίνακας Α έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 \neq \lambda_3$ έχουμε:

$$\text{Cos}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\text{Cos}[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{21} = \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}]\text{Cos}[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{31} = \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}]\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}]\text{Cos}[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{12} = \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}]\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}]\text{Cos}[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{22} = \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{32} = \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{13} = \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) - 2\lambda_1\alpha_{13}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{23} = \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + 2\lambda_1\alpha_{23}]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

$$p_{33} = \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{33} + \lambda_3(\lambda_3 - 2\lambda_1)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{33} + \lambda_1^2]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} -$$

$$\frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + \alpha_{33}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)}$$

Όμοια αν ο πίνακας Α έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 \neq \lambda_3$ έχουμε:

$$\tan[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{21} = \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{31} = \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$\begin{aligned}
p_{12} = & \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) - 2\lambda_1\alpha_{13}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{23}] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}
\end{aligned}$$

$$p_{33} = \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{33} + \lambda_3(\lambda_3 - 2\lambda_1)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{33} + \lambda_1^2]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + \alpha_{33}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

Όμοια αν ο πίνακας A έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 \neq \lambda_3$ έχουμε:

$$\text{Cot}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 - \cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{21} = \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}]\cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)](1 - \cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{31} = \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}]\cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}]\cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)](1 - \cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{12} = \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)](-1 - Cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{22} = \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](-1 - Cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{32} = \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](-1 - Cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{13} = \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) - 2\lambda_1\alpha_{13}]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)](-1 - Cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{23} = \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{23}]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)](-1 - Cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

$$p_{33} = \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{33} + \lambda_3(\lambda_3 - 2\lambda_1)]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{33} + \lambda_1^2]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)^2} +$$

$$\frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + \alpha_{33}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](-1 - Cot^2[\lambda_1])}{(\lambda_3 - \lambda_1)}$$

3^η περίπτωση:

Ο πίνακας Α έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 = \lambda_3$.

Δεδομένα αριθμητικής παρεμβολής

$$\{\lambda_1, \{Sin[\lambda_1], Cos[\lambda_1], -Sin[\lambda_1]\}\}$$

Πολυώνυμο παρεμβολής με τις ιδιότητες :

$$p(\lambda_1) = Sin[\lambda_1], p'(\lambda_1) = Cos[\lambda_1], p''(\lambda_1) = -Sin[\lambda_1]$$

Χρησιμοποιώντας την μέθοδο Hermite βρίσκω το πολυώνυμο παρεμβολής $p(x)$:

Hermite

X_i	$F[X_i]$	$F[X_i, X_{i+1}]$	$F[X_i, X_{i+1}, X_{i+2}]$
$X_0 = \lambda_1$	$\sin[\lambda_1]$		
		$\cos[\lambda_1]$	
$X_1 = \lambda_1$	$\sin[\lambda_1]$		$-\sin[\lambda_1]$
		$\cos[\lambda_1]$	
$X_2 = \lambda_1$	$\sin[\lambda_1]$		

$$F[x_0, x_1] = p'(x_0) = \cos[\lambda_1]$$

$$F[x_1, x_2] = p'(x_0) = \cos[\lambda_1]$$

$$F[x_0, x_1, x_2] = p''(x_0) = p''(\lambda_1) = -\sin[\lambda_1]$$

Το πολυώνυμο παρεμβολής προκύπτει από τον τύπο:

$$p(x) = F[x_0] + F[x_0, x_1](x - x_0) + F[x_0, x_1, x_2](x - x_0)(x - x_1) =$$

$$\sin[\lambda_1] + \cos[\lambda_1](x - \lambda_1) - \sin[\lambda_1](x - \lambda_1)(x - x_1) =$$

$$\sin[\lambda_1] + \cos[\lambda_1] - \lambda_1 \cos[\lambda_1] - x^2 \sin[\lambda_1] + 2\lambda_1 \sin[\lambda_1]x - \lambda_1^2 \sin[\lambda_1]$$

$$p(x) = -\sin[\lambda_1]x^2 + [\cos[\lambda_1] + 2\lambda_1 \sin[\lambda_1]]x + x + (1 - \lambda_1^2)\sin[\lambda_1] - \lambda_1 \cos[\lambda_1]$$

Ορίζω $\sin[A] = p(A)$ δηλαδή:

$$\sin[A] = p(A) = -\sin[\lambda_1]A^2 + [2\lambda_1 \sin[\lambda_1] + \cos[\lambda_1]]A + (1 - \lambda_1^2)\sin[\lambda_1] - \lambda_1 \cos[\lambda_1]I_3$$

και μετά από πράξεις έχουμε:

$$\text{Sin}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = [-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + [2\lambda_1\alpha_{11} + (1 - \lambda_1^2)]]\text{Sin}[\lambda_1] + (\alpha_{11} - \lambda_1)\text{Cos}[\lambda_1]$$

$$p_{21} = [-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\text{Sin}[\lambda_1] + \alpha_{21}\text{Cos}[\lambda_1]$$

$$p_{31} = [-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}]\text{Sin}[\lambda_1] + \alpha_{31}\text{Cos}[\lambda_1]$$

$$p_{12} = [-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}]\text{Sin}[\lambda_1] + \alpha_{12}\text{Cos}[\lambda_1]$$

$$p_{13} = [-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}]\text{Sin}[\lambda_1] + \alpha_{13}\text{Cos}[\lambda_1]$$

$$p_{22} = [-(a_{12}\alpha_{21} + \alpha_{23}\alpha_{32} + \alpha_{22}^2) + [2\lambda_1\alpha_{22} + (1 - \lambda_1^2)]]\text{Sin}[\lambda_1] + (\alpha_{22} - \lambda_1)\text{Cos}[\lambda_1]$$

$$p_{32} = [-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}]\text{Sin}[\lambda_1] + \alpha_{32}\text{Cos}[\lambda_1]$$

$$p_{23} = [-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + 2\lambda_1\alpha_{23}]\text{Sin}[\lambda_1] + \alpha_{23}\text{Cos}[\lambda_1]$$

$$p_{33} = [-(a_{13}\alpha_{31} + \alpha_{23}\alpha_{32} + \alpha_{33}^2) + [\lambda_1(2\alpha_{33} - \lambda_1) + 1]]\text{Sin}[\lambda_1] + (\alpha_{32} - \lambda_1)\text{Cos}[\lambda_1]$$

Όμοια όταν ο πίνακας A έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 = \lambda_3$ έχουμε:

$$\text{Cos}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = [-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + [2\lambda_1\alpha_{11} + (1 - \lambda_1^2)]]\text{Cos}[\lambda_1] - (\alpha_{11} - \lambda_1)\text{Sin}[\lambda_1]$$

$$p_{21} = [-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\text{Cos}[\lambda_1] - \alpha_{21}\text{Sin}[\lambda_1]$$

$$p_{31} = [-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}] \cos[\lambda_1] - \alpha_{31}\sin[\lambda_1]$$

$$p_{12} = [-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}] \cos[\lambda_1] - \alpha_{12}\sin[\lambda_1]$$

$$p_{13} = [-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}] \cos[\lambda_1] - \alpha_{13}\sin[\lambda_1]$$

$$p_{22} = [-(a_{12}\alpha_{21} + \alpha_{23}\alpha_{32} + \alpha_{22}^2) + [2\lambda_1\alpha_{22} + (1 - \lambda_1^2)]] \cos[\lambda_1] - (\alpha_{22} - \lambda_1)\sin[\lambda_1]$$

$$p_{32} = [-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}] \cos[\lambda_1] - \alpha_{32}\sin[\lambda_1]$$

$$p_{23} = [-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + 2\lambda_1\alpha_{23}] \cos[\lambda_1] - \alpha_{23}\sin[\lambda_1]$$

$$p_{33} = [-(a_{13}\alpha_{31} + \alpha_{23}\alpha_{32} + \alpha_{33}^2) + [\lambda_1(2\alpha_{33} - \lambda_1) + 1]] \cos[\lambda_1] - (\alpha_{32} - \lambda_1)\sin[\lambda_1]$$

Όμοια όταν ο πίνακας A έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 = \lambda_3$ έχουμε:

$$\tan[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = [(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + (\alpha_{11} - \lambda_1)(1 + \tan^2[\lambda_1]) + \tan[\lambda_1]$$

$$p_{21} = [(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + \alpha_{21}(1 + \tan^2[\lambda_1])$$

$$p_{31} = [(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + \alpha_{31}(1 + \tan^2[\lambda_1])$$

$$p_{12} = [(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + \alpha_{12}(1 + \tan^2[\lambda_1])$$

$$p_{13} = [(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) - 2\lambda_1\alpha_{13}][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + \alpha_{13}(1 + \tan^2[\lambda_1])$$

$$p_{22} = [(a_{12}\alpha_{21} + \alpha_{23}\alpha_{32} + \alpha_{22}^2) - [2\lambda_1\alpha_{22} + \lambda_1^2][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + (\alpha_{22} - \lambda_1)(1 + \tan^2[\lambda_1]) + \tan[\lambda_1]$$

$$p_{23} = [(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}][2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + \alpha_{23}(1 + \tan^2[\lambda_1])$$

$$p_{32} = [(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}] [2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + \alpha_{32}(1 + \tan^2[\lambda_1])$$

$$p_{33} = [(a_{13}\alpha_{31} + \alpha_{23}\alpha_{32} + \alpha_{33}^2) - [2\lambda_1\alpha_{33} + \lambda_1^2] [2\tan[\lambda_1](1 + \tan^2[\lambda_1])] + (\alpha_{32} - \lambda_1)(1 + \tan^2[\lambda_1]) + \tan[\lambda_1]$$

Όμοια όταν ο πίνακας Α έχει τρείς ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$ όπου $\lambda_1 = \lambda_2 = \lambda_3$ έχουμε:

$$\text{Cot}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

όπου:

$$p_{11} = [(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - (\alpha_{11} - \lambda_1)(1 + \cot^2[\lambda_1]) + \cot[\lambda_1]$$

$$p_{21} = [(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - \alpha_{21}(1 + \cot^2[\lambda_1])$$

$$p_{31} = [(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - \alpha_{31}(1 + \cot^2[\lambda_1])$$

$$p_{12} = [(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - \alpha_{12}(1 + \cot^2[\lambda_1])$$

$$p_{13} = [(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) - 2\lambda_1\alpha_{13}] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - \alpha_{13}(1 + \cot^2[\lambda_1])$$

$$p_{22} = [(a_{12}\alpha_{21} + \alpha_{23}\alpha_{32} + \alpha_{22}^2) - [2\lambda_1\alpha_{22} + \lambda_1^2] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - (\alpha_{22} - \lambda_1)(1 + \cot^2[\lambda_1]) + \cot[\lambda_1]$$

$$p_{23} = [(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - \alpha_{23}(1 + \cot^2[\lambda_1])$$

$$p_{32} = [(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - \alpha_{32}(1 + \cot^2[\lambda_1])$$

$$p_{33} = [(a_{13}\alpha_{31} + \alpha_{23}\alpha_{32} + \alpha_{33}^2) - [2\lambda_1\alpha_{33} + \lambda_1^2] [2\cot[\lambda_1](1 + \cot^2[\lambda_1])] - (\alpha_{32} - \lambda_1)(1 + \cot^2[\lambda_1]) + \cot[\lambda_1]$$

4.1.3. Ορισμός Πίνακα $\nu \times \nu$

Έστω $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ ένας $n \times n$ πίνακας. Ονομάζεται χαρακτηριστικό

πολυώνυμο του πίνακα αυτού το πολυώνυμο:

$$p(x) = \text{Det}(A - xI_n)$$

Συμβολίζω με $\lambda_1, \lambda_2, \dots, \lambda_n$ τις ρίζες του πολυωνύμου αυτού, οι οποίες λέγονται ιδιοτιμές του πίνακα A.

Ορισμός του πίνακα $\text{Sin}[A]$

Διακρίνω τις περιπτώσεις:

1^η περίπτωση:

Ο πίνακας A έχει η άνισες ιδιοτιμές $\lambda_1, \lambda_2, \dots, \lambda_n$. Όλες οι ιδιοτιμές του να είναι απλού βαθμού πολλαπλότητας.

Δεδομένα αριθμητικής παρεμβολής :

$$(\lambda_1, \text{Sin}[\lambda_1]), (\lambda_2, \text{Sin}[\lambda_2]), \dots, (\lambda_n, \text{Sin}[\lambda_n])$$

Χρησιμοποιώντας την μέθοδο Newton ή μέθοδο Διαιρεμένων Διαφορών (1.4)

βρίσκω το πολυώνυμο παρεμβολής $p(x)$.

$$\text{Ορίζω ως } \text{Sin}[A] = p(A).$$

Αντίστοιχα ορίζω :

Τον πίνακα $Cos[A]$ με δεδομένα αριθμητικής παρεμβολής :

$$(\lambda_1, Cos[\lambda_1]), (\lambda_2, Cos[\lambda_2]), \dots, (\lambda_n, Cos[\lambda_n])$$

Χρησιμοποιώντας την μέθοδο Newton ή μέθοδο Διαιρεμένων Διαφορών (1.4)

βρίσκω το πολυώνυμο παρεμβολής $p(x)$.

Ορίζω ως $Cos[A] = p(A)$.

Τον πίνακα $Tan[A]$ με δεδομένα αριθμητικής παρεμβολής :

$$(\lambda_1, Tan[\lambda_1]), (\lambda_2, Tan[\lambda_2]), \dots, (\lambda_n, Tan[\lambda_n])$$

Χρησιμοποιώντας την μέθοδο Newton ή μέθοδο Διαιρεμένων Διαφορών (1.4)

βρίσκω το πολυώνυμο παρεμβολής $p(x)$.

Ορίζω ως $Tan[A] = p(A)$.

Και τέλος τον πίνακα $Cot[A]$ με δεδομένα αριθμητικής παρεμβολής :

$$(\lambda_1, Cot[\lambda_1]), (\lambda_2, Cot[\lambda_2]), \dots, (\lambda_n, Cot[\lambda_n])$$

Χρησιμοποιώντας την μέθοδο Newton ή μέθοδο Διαιρεμένων Διαφορών (1.4)

βρίσκω το πολυώνυμο παρεμβολής $p(x)$.

Ορίζω ως $Cot[A] = p(A)$.

2^η περίπτωση:

Ο πίνακας Α έχει ιδιοτιμές με βαθμό **πολλαπλότητας** ≥ 2 .

Δεδομένα αριθμητικής παρεμβολής με τις ιδιότητες :

$p(x)$ και την $p'(x)$ όταν έχω ρίζα βαθμού πολλαπλότητας 2.

$p(x)$, την $p'(x)$ και την $p''(x)$, όταν έχω ρίζα βαθμού πολλαπλότητας 3.

$p(x)$, $p'(x)$ μέχρι και $p^{(n-1)}(x)$, όταν έχω ρίζα βαθμού πολλαπλότητας n.

Χρησιμοποιώντας την μέθοδο των Διαιρεμένων Διαφορών Hermite (1.5)

βρίσκω το πολυώνυμο παρεμβολής $p(x)$.

Ανάλογα με τη συνάρτηση την οποία έχω (Sin, Cos, Tan, Cot) ορίζω ως

$$\text{Sin}[A] = p(A)$$

$$\text{Cos}[A] = p(A)$$

$$\text{Tan}[A] = p(A)$$

$$\text{Cot}[A] = p(A)$$

4.2 Με δυναμοσειρές

- Γνωρίζουμε ότι η συνάρτηση $\text{Sin}[x]$ αναπτύσσεται στη δυναμοσειρά:

$$\text{Sin}[x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$$

για κάθε $x \in R$.

Αν A είναι ένας ηχη πίνακας, ορίζω ως $\text{Sin}[A]$ το :

$$\sin[A] = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \frac{A^9}{9!} - \frac{A^{11}}{11!} + \dots$$

Αποδεικνύεται ότι το άθροισμα του 2ου μέλους υπάρχει πάντα.

- Γνωρίζουμε ότι η συνάρτηση $\cos[x]$ αναπτύσσεται στη δυναμοσειρά:

$$\cos[x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{x^{2k}}{(2k)!}$$

για κάθε $x \in R$.

Αν A είναι ένας οχη πίνακας, ορίζω ως $\cos[A]$ το :

$$\cos[A] = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \frac{A^8}{8!} - \frac{A^{10}}{10!} + \dots$$

Αποδεικνύεται ότι το άθροισμα του 2ου μέλους υπάρχει πάντα.

- Γνωρίζουμε ότι η συνάρτηση $\tan[x]$ αναπτύσσεται στη δυναμοσειρά:

$$\tan[x] = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots = \sum_{k=1}^{+\infty} \frac{B_k 2^{2k} (2^{2k}-1)}{(2k)!} x^{2k-1}$$

για κάθε $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Αν A είναι ένας οχη πίνακας, ορίζω ως $\tan[A]$ το :

$$\tan[A] = A + \frac{A^3}{3} + \frac{2A^5}{15} + \frac{17A^7}{315} + \dots = \sum_{k=1}^{+\infty} \frac{B_k 2^{2k} (2^{2k}-1)}{(2k)!} A^{2k-1}$$

Το άθροισμα του 2ου μέλους υπάρχει υπό προϋποθέσεις.

- Γνωρίζουμε ότι η συνάρτηση $\cot[x]$ αναπτύσσεται στη δυναμοσειρά:

$$\cot[x] = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots = \sum_{k=1}^{+\infty} \frac{B_k 2^{2k}}{(2k)!} x^{2k-1}$$

για κάθε $x \in (0, \pi)$.

Αν A είναι ένας ηχη πίνακας, ορίζω ως $\text{Cot}[A]$ το :

$$\text{Cot}[A] = A^{-1} - \frac{A}{3} - \frac{A^3}{45} - \frac{2A^5}{945} - \dots = \sum_{k=1}^{+\infty} \frac{B_k 2^{2k}}{(2k)!} A^{2k-1}$$

Το άθροισμα του 2^{ου} μέλους υπάρχει υπό προϋποθέσεις.

Όπου B_n είναι οι αριθμοί του Bernoulli και είναι:

$$B_n = \frac{(2n)!}{2^{2n-1} \pi^{2n}} \left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \dots \right)$$

$$\text{ή προσεγγιστικά } B_n \approx 4 \left(\frac{n}{\pi e} \right)^{2n} \sqrt{\pi n}$$

4.3 Τριγωνομετρικές Ταυτότητες

$$\text{Sin}^2[A] + \text{Cos}^2[A] = I$$

$$\text{Tan}[A] = \text{Sin}[A] \text{Cos}[A]^{-1} = \frac{\text{Sin}[A]}{\text{Cos}[A]}$$

$$\text{Cot}[A] = \text{Cos}[A] \text{Sin}[A]^{-1} = \frac{\text{Cos}[A]}{\text{Sin}[A]}$$

$$I + \text{Tan}^2[A] = (\text{Cos}^2[A])^{-1} = \frac{1}{\text{Cos}^2[A]}$$

$$I + \text{Cot}^2[A] = (\text{Sin}^2[A])^{-1} = \frac{1}{\text{Sin}^2[A]}$$

Ταυτότητες που αναφέρονται στους τριγωνομετρικούς αριθμούς της «γωνίας» $2A$

$$\text{Sin}[2A] = 2 \text{Sin}[A] \text{Cos}[A]$$

$$\cos[2A] = \cos^2[A] - \sin^2[A]$$

$$\tan[2A] = \frac{2\tan[A]}{1 - \tan^2[A]}$$

$$\cot[2A] = \frac{\cot^2[A] - 1}{2\cot[A]}$$

Ταυτότητες που αναφέρονται στους τριγωνομετρικούς αριθμούς του αθροίσματος $A+B$, αν και μόνο αν $AB=BA$:

$$\sin[A+B] = \sin[A]\cos[B] + \cos[A]\sin[B]$$

$$\cos[A+B] = \cos[A]\cos[B] - \sin[A]\sin[B]$$

$$\tan[A+B] = \frac{\tan[A]+\tan[B]}{1-\tan[A]\tan[B]}$$

$$\cot[A+B] = \frac{\cot[A]\cot[B]-1}{\cot[A]+\cot[B]}$$

4.4 Παραδείγματα

1° παράδειγμα:

Έχουμε τον πίνακα $A = \begin{pmatrix} \frac{\pi}{6} & -\frac{\pi}{12} & -\frac{\pi}{12} \\ \frac{\pi}{6} & \frac{5\pi}{12} & \frac{\pi}{12} \\ -\frac{\pi}{6} & -\frac{\pi}{6} & \frac{\pi}{6} \end{pmatrix}$, με ιδιοτιμές $\lambda_1 = \frac{\pi}{3}, \lambda_2 = \frac{\pi}{4}, \lambda_3 = \frac{\pi}{6}$

και με:

$$A^2 = \begin{pmatrix} \frac{\pi^2}{36} & -\frac{5\pi^2}{144} & -\frac{5\pi^2}{144} \\ \frac{\pi^2}{12} & \frac{7\pi^2}{48} & \frac{5\pi^2}{144} \\ -\frac{\pi^2}{12} & -\frac{\pi^2}{12} & \frac{\pi^2}{36} \end{pmatrix}$$

➤ Βρίσκουμε το $\text{Sin}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
 p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
 &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Sin}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
 &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} - \\
 &= \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)\text{Sin}[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
 &\quad + \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)\text{Sin}[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
 &\quad + \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)\text{Sin}[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
p_{21} = & \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
= & \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\sin[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\sin[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\sin[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}(-1 + \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
= & \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\sin[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\sin[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\sin[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}(1 - \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\sin[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\sin[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\sin[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2} - \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\sin[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\sin[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\sin[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2} - \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)\sin[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)\sin[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)\sin[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}(-1 + \sqrt{2})
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\sin[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\sin[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\sin[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = -\frac{1}{2} + \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{6} - \frac{\pi}{4}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\sin[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{3} - \frac{\pi}{6}) - \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\sin[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\sin[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2}(1 - \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{33} = & \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{6} - \frac{\pi}{4}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}) + \frac{\pi^2}{24}(\frac{\pi}{6} - \frac{\pi}{4}))\sin[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{3} - \frac{\pi}{6}) + \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}) + \frac{\pi}{3}\frac{\pi}{6}(\frac{\pi}{3} - \frac{\pi}{6}))\sin[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{4} - \frac{\pi}{3}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}) + \frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}))\sin[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$Sin[A] = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} - \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{1}{\sqrt{2}} \\ \frac{1}{2}(-1 + \sqrt{3}) & \frac{1}{2}(-1 + \sqrt{2} + \sqrt{3}) & -\frac{1}{2} + \frac{1}{\sqrt{2}} \\ \frac{1}{2}(1 - \sqrt{3}) & \frac{1}{2}(1 - \sqrt{3}) & \frac{1}{2} \end{pmatrix}$$

➤ Βρίσκουμε το $Cos[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned} p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]Cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]Cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]Cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\ &= \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)Cos[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\ &\quad \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)Cos[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\ &\quad \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)Cos[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned}
p_{21} = & \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\cos[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\cos[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\cos[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}(1 - \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\cos[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\cos[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\cos[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}(-1 + \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\cos[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\cos[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\cos[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2}(-\sqrt{2} + \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\cos[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\cos[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\cos[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2}(-\sqrt{2} + \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)\cos[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)\cos[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)\cos[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{2}(1 + \sqrt{2} - \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\cos[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\cos[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\cos[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{6} - \frac{\pi}{4}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\cos[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{3} - \frac{\pi}{6}) - \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\cos[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\cos[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{2}(-1 + \sqrt{3})
\end{aligned}$$

$$\begin{aligned}
p_{33} = & \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{6} - \frac{\pi}{4}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}) + \frac{\pi^2}{24}(\frac{\pi}{6} - \frac{\pi}{4}))\cos[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{3} - \frac{\pi}{6}) + \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}) + \frac{\pi}{3}\frac{\pi}{6}(\frac{\pi}{3} - \frac{\pi}{6}))\cos[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{4} - \frac{\pi}{3}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}) + \frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}))\cos[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{\sqrt{3}}{2}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$Cos[A] = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}(-\sqrt{2} + \sqrt{3}) & \frac{1}{2}(-\sqrt{2} + \sqrt{3}) \\ \frac{1}{2}(1 - \sqrt{3}) & \frac{1}{2}(1 + \sqrt{2} - \sqrt{3}) & \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \\ \frac{1}{2}(-1 + \sqrt{3}) & \frac{1}{2}(-1 + \sqrt{3}) & \frac{\sqrt{3}}{2} \end{pmatrix}$$

➤ Βρίσκουμε τον πίνακα $Tan[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned} p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]Tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]Tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]Tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\ &= \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)Tan[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\ &\quad \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)Tan[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\ &\quad \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)Tan[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned}
p_{21} &= \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&+ \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\tan[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
&+ \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\tan[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
&+ \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\tan[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{31} &= \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&+ \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&+ \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\tan[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
&+ \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\tan[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
&+ \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\tan[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = -\frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\tan[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\tan[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\tan[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = -1 + \frac{1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\tan[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\tan[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\tan[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = -1 + \frac{1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)\tan[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)\tan[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)\tan[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = 1 + \frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\tan[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\tan[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\tan[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = 1 - \frac{1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{6} - \frac{\pi}{4}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\tan[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{3} - \frac{\pi}{6}) - \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\tan[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\tan[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = -\frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{33} = & \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{6} - \frac{\pi}{4}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}) + \frac{\pi^2}{24}(\frac{\pi}{6} - \frac{\pi}{4}))\tan[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{3} - \frac{\pi}{6}) + \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}) + \frac{\pi}{3}\frac{\pi}{6}(\frac{\pi}{3} - \frac{\pi}{6}))\tan[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{4} - \frac{\pi}{3}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}) + \frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}))\tan[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{1}{\sqrt{3}}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$Tan[A] = \begin{pmatrix} \frac{1}{\sqrt{3}} & -1 + \frac{1}{\sqrt{3}} & -1 + \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & 1 + \frac{2}{\sqrt{3}} & 1 - \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

➤ Τέλος ο πίνακας $Cot[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$, κάνοντας χρήση των τύπων

που αποδείξαμε είναι:

$$\begin{aligned} p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]Cot[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]Cot[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\ &\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]Cot[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\ &= \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)Cot[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\ &\quad \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)Cot[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\ &\quad \frac{\left(\frac{\pi^2}{36}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)Cot[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \sqrt{3} \end{aligned}$$

$$\begin{aligned}
p_{21} = & \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\text{Cot}\left[\frac{\pi}{3}\right]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\text{Cot}\left[\frac{\pi}{4}\right]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\text{Cot}\left[\frac{\pi}{6}\right]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = -\frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\text{Cot}\left[\frac{\pi}{3}\right]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \frac{\pi}{6}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\text{Cot}\left[\frac{\pi}{4}\right]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& + \frac{\left(-\frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + \frac{\pi}{6}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\text{Cot}\left[\frac{\pi}{6}\right]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\text{Cot}[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\text{Cot}[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\text{Cot}[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = -1 + \sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{6} - \frac{\pi}{4}) + (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\text{Cot}[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{3} - \frac{\pi}{6}) + (-\frac{\pi}{12})(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\text{Cot}[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{5\pi^2}{144}(\frac{\pi}{4} - \frac{\pi}{3}) - (-\frac{\pi}{12})(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\text{Cot}[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = -1 + \sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right) + \frac{\pi^2}{24}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right)\text{Cot}[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{5\pi}{12}\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right) + \frac{\pi}{3}\frac{\pi}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)\text{Cot}[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{7\pi^2}{48}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \frac{5\pi}{12}\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right) + \frac{\pi^2}{12}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)\text{Cot}[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = 1 - \frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{36}\right)\right)\text{Cot}[\frac{\pi}{3}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{36} - \frac{\pi^2}{9}\right)\right)\text{Cot}[\frac{\pi}{4}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} + \\
& \frac{\left(\frac{5\pi^2}{144}\left(\frac{\pi}{4} - \frac{\pi}{3}\right) - \left(\frac{\pi}{12}\right)\left(\frac{\pi^2}{16} - \frac{\pi^2}{9}\right)\right)\text{Cot}[\frac{\pi}{6}]}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = 1 - \sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{6} - \frac{\pi}{4}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}))\text{Cot}[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{3} - \frac{\pi}{6}) - \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}))\text{Cot}[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(-\frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}))\text{Cot}[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{2}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
p_{33} = & \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3]\text{Cot}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Cot}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Cot}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{6} - \frac{\pi}{4}) + \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{36}) + \frac{\pi^2}{24}(\frac{\pi}{6} - \frac{\pi}{4}))\text{Cot}[\frac{\pi}{3}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{3} - \frac{\pi}{6}) + \frac{\pi}{6}(\frac{\pi^2}{36} - \frac{\pi^2}{9}) + \frac{\pi}{3}\frac{\pi}{6}(\frac{\pi}{3} - \frac{\pi}{6}))\text{Cot}[\frac{\pi}{4}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} + \\
& \frac{(\frac{\pi^2}{36}(\frac{\pi}{4} - \frac{\pi}{3}) - \frac{\pi}{6}(\frac{\pi^2}{16} - \frac{\pi^2}{9}) + \frac{\pi^2}{12}(\frac{\pi}{4} - \frac{\pi}{3}))\text{Cot}[\frac{\pi}{6}]}{(\frac{\pi}{6} - \frac{\pi}{3})(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{3})} = \sqrt{3}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$Cot[A] = \begin{pmatrix} \sqrt{3} & -1+\sqrt{3} & -1+\sqrt{3} \\ -\frac{2}{\sqrt{3}} & 1-\frac{2}{\sqrt{3}} & 1-\sqrt{3} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \sqrt{3} \end{pmatrix}$$

2^ο παράδειγμα:

Έχουμε τον πίνακα $A = \begin{pmatrix} -\frac{4\pi}{3} & -\frac{2\pi}{3} & \frac{\pi}{3} \\ \frac{5\pi}{3} & \pi & -\frac{\pi}{3} \\ -\frac{10\pi}{3} & -\frac{4\pi}{3} & \pi \end{pmatrix}$, με ιδιοτιμές $\lambda_1 = \frac{\pi}{3}, \lambda_2 = \frac{\pi}{3}, \lambda_3 = 0$

και με:

$$A^2 = \begin{pmatrix} -\frac{4\pi^2}{9} & -\frac{2\pi^2}{9} & \frac{\pi^2}{9} \\ \frac{5\pi^2}{9} & \frac{\pi^2}{3} & -\frac{\pi^2}{9} \\ -\frac{10\pi^2}{9} & -\frac{4\pi^2}{9} & \frac{\pi^2}{3} \end{pmatrix}$$

➤ Βρίσκουμε το $Sin[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
p_{11} = & \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{4\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{4\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{4\pi^2}{9} - 2\frac{\pi}{3}(-\frac{4\pi}{3}) + \frac{\pi^2}{9}]\sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{4\pi^2}{9}) + (-\frac{4\pi}{3})(\frac{\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -2\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{21} = & \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(\frac{5\pi^2}{9}) + 2\frac{\pi}{3}(\frac{5\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{5\pi^2}{9} - 2\frac{\pi}{3}(\frac{5\pi}{3}) + \frac{\pi^2}{9}]\sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(\frac{5\pi^2}{9}) + (\frac{5\pi}{3})(\frac{\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = \frac{5\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{10\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{10\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{10\pi^2}{9} - 2\frac{\pi}{3}(-\frac{10\pi}{3}) + \frac{\pi^2}{9}] \sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{10\pi^2}{9}) + (-\frac{10\pi}{3})(\frac{\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -5\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{2\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{2\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{2\pi^2}{9} - 2\frac{\pi}{3}(-\frac{2\pi}{3}) + \frac{\pi^2}{9}] \sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{2\pi^2}{9}) + (-\frac{2\pi}{3})(\frac{\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(\frac{\pi^2}{3}) + 2\frac{\pi}{3}\pi]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{\pi^2}{3} - 2\frac{\pi}{3}\pi + \frac{\pi^2}{9}]\sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(\frac{\pi^2}{3}) + \pi(\frac{\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = \frac{3\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(\frac{4\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{4\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{4\pi^2}{9} - 2\frac{\pi}{3}(-\frac{4\pi}{3}) + \frac{\pi^2}{9}]\sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(\frac{4\pi^2}{9}) + (-\frac{4\pi}{3})(\frac{\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -2\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{13}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
& = \frac{[-(\frac{\pi^2}{9}) + 2\frac{\pi}{3}(\frac{\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{\pi^2}{9} - 2\frac{\pi}{3}(\frac{\pi}{3}) + \frac{\pi^2}{9}] \sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(\frac{\pi^2}{9}) + (\frac{\pi}{3})(\frac{\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = \frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{23}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
& = \frac{[-(-\frac{\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{\pi^2}{9} - 2\frac{\pi}{3}(-\frac{\pi}{3}) + \frac{\pi^2}{9}] \sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{\pi^2}{9}) + (\frac{\pi}{3})(-\frac{\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -\frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
p_{33} &= \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{33} + \lambda_3(\lambda_3 - 2\lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{33} + \lambda_1^2]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + \alpha_{33}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(-\frac{\pi^2}{3}) + 2\frac{\pi}{3}(\pi)]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{\pi^2}{3} - 2\frac{\pi}{3}(\pi) + \frac{\pi^2}{9}]\sin[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(-\frac{\pi^2}{3}) + (\frac{\pi}{3})(\pi)]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = \frac{3\sqrt{3}}{2}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$\sin[A] = \begin{pmatrix} -2\sqrt{3} & -\sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{5\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ -5\sqrt{3} & -2\sqrt{3} & \frac{3\sqrt{3}}{2} \end{pmatrix}$$

➤ Βρίσκουμε το $\cos[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
p_{11} = & \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{4\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{4\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{4\pi^2}{9} - 2\frac{\pi}{3}(-\frac{4\pi}{3}) + \frac{\pi^2}{9}]\cos[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{4\pi^2}{9}) + (-\frac{4\pi}{3})(\frac{\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = 3
\end{aligned}$$

$$\begin{aligned}
p_{21} = & \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(\frac{5\pi^2}{9}) + 2\frac{\pi}{3}(\frac{5\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{5\pi^2}{9} - 2\frac{\pi}{3}(\frac{5\pi}{3})]\cos[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(\frac{5\pi^2}{9}) + (\frac{5\pi}{3})(\frac{\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -\frac{5}{2}
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[-(a_{11}\alpha_{31} + a_{21}\alpha_{32} + a_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{31} + a_{21}\alpha_{32} + a_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}] \cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{11}\alpha_{31} + a_{21}\alpha_{32} + a_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{10\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{10\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{10\pi^2}{9} - 2\frac{\pi}{3}(-\frac{10\pi}{3})] \cos[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{10\pi^2}{9}) + (-\frac{10\pi}{3})(\frac{\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = 5
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[-(a_{11}\alpha_{12} + a_{12}\alpha_{22} + a_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{11}\alpha_{12} + a_{12}\alpha_{22} + a_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}] \cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{11}\alpha_{12} + a_{12}\alpha_{22} + a_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{2\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{2\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{2\pi^2}{9} - 2\frac{\pi}{3}(-\frac{2\pi}{3})] \cos[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{2\pi^2}{9}) + (-\frac{2\pi}{3})(\frac{\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = 1
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(\frac{\pi^2}{3}) + 2\frac{\pi}{3}\pi]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{\pi^2}{3} - 2\frac{\pi}{3}\pi + \frac{\pi^2}{9}]\cos[0]}{(-\frac{\pi}{3})^2} + \\
+ & \frac{[-(\frac{\pi^2}{3}) + \pi(\frac{\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{4\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{4\pi}{3})]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{4\pi^2}{9} - 2\frac{\pi}{3}(-\frac{4\pi}{3})]\cos[0]}{(-\frac{\pi}{3})^2} + \\
+ & \frac{[-(-\frac{4\pi^2}{9}) + (-\frac{4\pi}{3})(\frac{\pi}{3})]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = 2
\end{aligned}$$

$$\begin{aligned}
p_{13} &= \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{13}] \cos[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
&- \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(\frac{\pi^2}{9}) + 2\frac{\pi}{3}(\frac{\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{\pi^2}{9} - 2\frac{\pi}{3}(\frac{\pi}{3})] \cos[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(\frac{\pi^2}{9}) + (\frac{\pi}{3})(\frac{\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -\frac{1}{2} \\
p_{23} &= \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{23}] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
&- \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(-\frac{\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{\pi}{3})] \sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{\pi^2}{9} - 2\frac{\pi}{3}(-\frac{\pi}{3})] \sin[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(-\frac{\pi^2}{9}) + (\frac{\pi}{3})(-\frac{\pi}{3})] \cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
p_{33} = & \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{33} + \lambda_3(\lambda_3 - 2\lambda_1)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{33} + \lambda_1^2]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)^2} - \\
& - \frac{[-(a_{13}\alpha_{31} + \alpha_{33}^2 + \alpha_{23}\alpha_{32}) + \alpha_{33}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{\pi^2}{3}) + 2\frac{\pi}{3}(\pi)]\sin[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{\pi^2}{3} - 2\frac{\pi}{3}(\pi) + \frac{\pi^2}{9}]\sin[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{\pi^2}{3}) + (\frac{\pi}{3})(\pi)]\cos[\frac{\pi}{3}]}{(-\frac{\pi}{3})} = -\frac{1}{2}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$\cos[A] = \begin{pmatrix} 3 & 1 & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{1}{2} & \frac{1}{2} \\ 5 & 2 & -\frac{1}{2} \end{pmatrix}$$

➤ Βρίσκουμε τον πίνακα $\tan[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
p_{11} &= \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + 2\lambda_1\alpha_{11} + \lambda_3(\lambda_3 - 2\lambda_1)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) - 2\lambda_1\alpha_{11} + \lambda_1^2]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{11}^2 + \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31}) + \alpha_{11}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(-\frac{4\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{4\pi}{3})]\tan[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{4\pi^2}{9} - 2\frac{\pi}{3}(-\frac{4\pi}{3}) + \frac{\pi^2}{9}]\tan[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(-\frac{4\pi^2}{9}) + (-\frac{4\pi}{3})(\frac{\pi}{3})](1 + \tan^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = -4\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{21} &= \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + 2\lambda_1\alpha_{21}]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) - 2\lambda_1\alpha_{21}]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{11}\alpha_{21} + \alpha_{21}\alpha_{22} + \alpha_{23}\alpha_{31}) + \alpha_{21}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(\frac{5\pi^2}{9}) + 2\frac{\pi}{3}(\frac{5\pi}{3})]\tan[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{5\pi^2}{9} - 2\frac{\pi}{3}(\frac{5\pi}{3})]\tan[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(\frac{5\pi^2}{9}) + (\frac{5\pi}{3})(\frac{\pi}{3})](1 + \tan^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = 5\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{31} &= \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + 2\lambda_1\alpha_{31}] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) - 2\lambda_1\alpha_{31}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{11}\alpha_{31} + \alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{33}) + \alpha_{31}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(-\frac{10\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{10\pi}{3})] \tan[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{10\pi^2}{9} - 2\frac{\pi}{3}(-\frac{10\pi}{3})] \tan[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(-\frac{10\pi^2}{9}) + (-\frac{10\pi}{3})(\frac{\pi}{3})](1 + \tan^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = -10\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{12} &= \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + 2\lambda_1\alpha_{12}] \tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{12}] \tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{32}) + \alpha_{12}(\lambda_3 + \lambda_1)](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(-\frac{2\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{2\pi}{3})] \tan[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{2\pi^2}{9} - 2\frac{\pi}{3}(-\frac{2\pi}{3})] \tan[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(-\frac{2\pi^2}{9}) + (-\frac{2\pi}{3})(\frac{\pi}{3})](1 + \tan^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = -2\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + 2\lambda_1\alpha_{21} + \lambda_3(\lambda_3 - 2\lambda_1)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) - 2\lambda_1\alpha_{22}]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{12}\alpha_{21} + \alpha_{22}^2 + \alpha_{23}\alpha_{32}) + \alpha_{22}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(\frac{\pi^2}{3}) + 2\frac{\pi}{3}\pi]\tan[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{\pi^2}{3} - 2\frac{\pi}{3}\pi + \frac{\pi^2}{9}]\tan[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(\frac{\pi^2}{3}) + \pi(\frac{\pi}{3})](1 + \tan^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = 3\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{32}]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) - 2\lambda_1\alpha_{32}]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
& + \frac{[-(a_{12}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{32}\alpha_{33}) + \alpha_{32}(\lambda_3 + \lambda_1) - \lambda_1\lambda_3](1 + \tan^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
= & \frac{[-(-\frac{4\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{4\pi}{3})]\tan[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{4\pi^2}{9} - 2\frac{\pi}{3}(-\frac{4\pi}{3})]\tan[0]}{(-\frac{\pi}{3})^2} + \\
& + \frac{[-(-\frac{4\pi^2}{9}) + (-\frac{4\pi}{3})(\frac{\pi}{3})](1 + \tan^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = -4\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{13} &= \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + 2\lambda_1\alpha_{13}] \operatorname{Tan}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{32}) - 2\lambda_1\alpha_{13}] \operatorname{Tan}[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33}) + \alpha_{13}(\lambda_3 + \lambda_1)](1 + \operatorname{Tan}^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(\frac{\pi^2}{9}) + 2\frac{\pi}{3}(\frac{\pi}{3})] \operatorname{Tan}[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[\frac{\pi^2}{9} - 2\frac{\pi}{3}(\frac{\pi}{3})] \operatorname{Tan}[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(\frac{\pi^2}{9}) + (\frac{\pi}{3})(\frac{\pi}{3})](1 + \operatorname{Tan}^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = \sqrt{3}
\end{aligned}$$

$$\begin{aligned}
p_{23} &= \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}) + 2\lambda_1\alpha_{23}] \operatorname{Tan}[\lambda_1]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) - 2\lambda_1\alpha_{23}] \operatorname{Tan}[\lambda_3]}{(\lambda_3 - \lambda_1)^2} + \\
&+ \frac{[-(a_{13}\alpha_{21} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) + \alpha_{23}(\lambda_3 + \lambda_1)](1 + \operatorname{Tan}^2[\lambda_1])}{(\lambda_3 - \lambda_1)} = \\
&= \frac{[-(-\frac{\pi^2}{9}) + 2\frac{\pi}{3}(-\frac{\pi}{3})] \operatorname{Tan}[\frac{\pi}{3}]}{(-\frac{\pi}{3})^2} + \frac{[-\frac{\pi^2}{9} - 2\frac{\pi}{3}(-\frac{\pi}{3})] \operatorname{Tan}[0]}{(-\frac{\pi}{3})^2} + \\
&+ \frac{[-(-\frac{\pi^2}{9}) + (\frac{\pi}{3})(-\frac{\pi}{3})](1 + \operatorname{Tan}^2[\frac{\pi}{3}])}{(-\frac{\pi}{3})} = -\sqrt{3}
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$\operatorname{Tan}[A] = \begin{pmatrix} -4\sqrt{3} & -2\sqrt{3} & \sqrt{3} \\ 5\sqrt{3} & 3\sqrt{3} & -\sqrt{3} \\ -10\sqrt{3} & -4\sqrt{3} & 3\sqrt{3} \end{pmatrix}$$

➤ Τέλος ο πίνακας $Cot[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ δεν ορίζεται αφού έχουμε

$\text{Sin}[0]=0$, συνεπώς δεν ορίζεται το $\text{Cot}[0]$.

3^ο παράδειγμα:

Έχουμε τον πίνακα $A = \begin{pmatrix} -\pi & 0 & -i\pi \\ i\pi & (-1-i)\pi & 3i\pi \\ 0 & 0 & (-1+i)\pi \end{pmatrix}$, με ιδιοτιμές

$$\lambda_1 = (-1+i)\pi, \lambda_2 = (-1-i)\pi, \lambda_3 = -\pi$$

και με:

$$A^2 = \begin{pmatrix} -\pi^2 & 0 & (1+2i)\pi^2 \\ (1-2i)\pi^2 & 2i\pi^2 & (1-6i)\pi^2 \\ 0 & 0 & -2i\pi^2 \end{pmatrix}$$

➤ Βρίσκουμε το $\text{Sin}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Sin}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{(\pi^2(-\pi - ((-1-i)\pi)) + (-\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi))\text{Sin}[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((\pi^2((-1+i)\pi + \pi)) + (-\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi))\text{Sin}[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(\pi^2((-1-i)\pi - (-1+i)\pi)) + (-\pi)((-1+i)\pi)^2)\text{Sin}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(-((-1-i)\pi)^2) + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi))\text{Sin}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{21} &= \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\text{Sin}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad + \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\text{Sin}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\text{Sin}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{((1-2i)\pi^2(-\pi - ((-1-i)\pi)) + (i\pi)((-1-i)\pi)^2 - \pi^2))\text{Sin}[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((1-2i)\pi^2((-1+i)\pi + \pi)) + (i\pi)(\pi^2 - ((-1+i)\pi)^2))\text{Sin}[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((1-2i)\pi^2((-1-i)\pi - (-1+i)\pi)) + (i\pi)((-1-i)\pi)^2 - ((-1+i)\pi)^2))\text{Sin}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -i\text{Sinh}[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
= & \frac{(0(-\pi - ((-1-i)\pi) + (0)((-1-i)\pi)^2 - \pi^2))\sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
= & \frac{(0(-\pi - ((-1-i)\pi) + (0)((-1-i)\pi)^2 - \pi^2))\sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{((1+2i)\pi^2(-\pi - ((-1-i)\pi) + (-i\pi)((-1-i)\pi)^2 - \pi^2))\sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1+2i)\pi^2((-1+i)\pi + \pi) + (-i\pi)(\pi^2 - ((-1+i)\pi)^2))\sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1+2i)\pi^2((-1-i)\pi - (-1+i)\pi) + (-i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = i\sinh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(2i\pi^2(-\pi - ((-1-i)\pi) + ((-1-i)\pi)((-1-i)\pi)^2 - \pi^2) + (-1-i)\pi^2)(-\pi - (-1-i)\pi))\sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(2i\pi^2((-1+i)\pi + \pi) + ((-1-i)\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi))\sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(2i\pi^2((-1-i)\pi - (-1+i)\pi) + ((-1-i)\pi)((-1+i)\pi)^2)\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(-(-1-i)\pi)^2 + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi))\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = i\sinh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{((1-6i)\pi^2(-\pi - ((-1-i)\pi)) + (3i\pi)((-1-i)\pi)^2 - \pi^2))\sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1-6i)\pi^2((-1+i)\pi + \pi) + (3i\pi)(\pi^2 - ((-1+i)\pi)^2))\sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1-6i)\pi^2((-1-i)\pi - (-1+i)\pi) + (3i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -3i\sinh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(0(-\pi - ((-1-i)\pi)) + (0)((-1-i)\pi)^2 - \pi^2))\sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{33} &= \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3] \sin[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)] \sin[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)] \sin[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{(-2i\pi^2(-\pi - ((-1-i)\pi) + ((-1+i)\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi)) \sin[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((-2i\pi^2((-1+i)\pi + \pi) + ((-1+i)\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi)) \sin[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(-2i\pi^2((-1-i)\pi - (-1+i)\pi) + ((-1+i)\pi)((-1+i)\pi)^2) \sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(-((1-i)\pi)^2) + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi)) \sin[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -i \sin[\pi]
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$\sin[A] = \begin{pmatrix} 0 & 0 & i \sinh[\pi] \\ -i \sinh[\pi] & i \sinh[\pi] & -3i \sinh[\pi] \\ 0 & 0 & -i \sinh[\pi] \end{pmatrix}$$

➤ Βρίσκουμε το $\text{Cos}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\text{Cos}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\text{Cos}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{(\pi^2(-\pi - ((-1-i)\pi)) + (-\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi))\text{Cos}[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((\pi^2((-1+i)\pi + \pi)) + (-\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi))\text{Cos}[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(\pi^2((-1-i)\pi - (-1+i)\pi)) + (-\pi)((-1+i)\pi)^2)\text{Cos}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(-((-1-i)\pi)^2) + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi))\text{Cos}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -1
\end{aligned}$$

$$\begin{aligned}
p_{21} &= \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]\text{Cos}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad + \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]\text{Cos}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]\text{Cos}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{((1-2i)\pi^2(-\pi - ((-1-i)\pi)) + (i\pi)((-1-i)\pi)^2 - \pi^2))\text{Cos}[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((1-2i)\pi^2((-1+i)\pi + \pi)) + (i\pi)(\pi^2 - ((-1+i)\pi)^2))\text{Cos}[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((1-2i)\pi^2((-1-i)\pi - (-1+i)\pi)) + (i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\text{Cos}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -1 + \text{Cosh}[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{(0(-\pi - ((-1-i)\pi) + (0)((-1-i)\pi)^2 - \pi^2))\cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{(0(-\pi - ((-1-i)\pi) + (0)((-1-i)\pi)^2 - \pi^2))\cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{((1+2i)\pi^2(-\pi - ((-1-i)\pi) + (-i\pi)((-1-i)\pi)^2 - \pi^2))\cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{((1+2i)\pi^2((-1+i)\pi + \pi) + (-i\pi)(\pi^2 - ((-1+i)\pi)^2))\cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{((1+2i)\pi^2((-1-i)\pi - (-1+i)\pi) + (-i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -1 + \cosh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(2i\pi^2(-\pi - ((-1-i)\pi) + ((-1-i)\pi)((-1-i)\pi)^2 - \pi^2) + (-1-i)\pi^2)(-\pi - (-1-i)\pi))\cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(2i\pi^2((-1+i)\pi + \pi) + ((-1-i)\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi))\cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(2i\pi^2((-1-i)\pi - (-1+i)\pi) + ((-1-i)\pi)((-1+i)\pi)^2)\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(-((1-i)\pi)^2) + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi))\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -\cosh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{((1-6i)\pi^2(-\pi - ((-1-i)\pi)) + (3i\pi)((-1-i)\pi)^2 - \pi^2))\cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1-6i)\pi^2((-1+i)\pi + \pi) + (3i\pi)(\pi^2 - ((-1+i)\pi)^2))\cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1-6i)\pi^2((-1-i)\pi - (-1+i)\pi) + (3i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -1\cosh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(0(-\pi - ((-1-i)\pi)) + (0)((-1-i)\pi)^2 - \pi^2))\cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{33} = & \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3] \cos[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)] \cos[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)] \cos[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(-2i\pi^2(-\pi - ((-1-i)\pi) + ((-1+i)\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi)) \cos[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{((-2i\pi^2((-1+i)\pi + \pi) + ((-1+i)\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi)) \cos[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(-2i\pi^2((-1-i)\pi - (-1+i)\pi) + ((-1+i)\pi)((-1+i)\pi)^2) \cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} - \\
& - \frac{(((1-i)\pi)^2) + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi)) \cos[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -\cosh[\pi]
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$\cos[A] = \begin{pmatrix} -1 & 0 & -1 + \cosh[\pi] \\ -1 + \cosh[\pi] & -\cosh[\pi] & -1 + \cosh[\pi] \\ 0 & 0 & -\cosh[\pi] \end{pmatrix}$$

➤ Βρίσκουμε τον πίνακα $Tan[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ κάνοντας χρήση των τύπων που αποδείξαμε.

$$\begin{aligned}
p_{11} &= \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_3 - \lambda_2) + a_{11}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]Tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_1 - \lambda_3) + a_{11}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]Tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31})(\lambda_2 - \lambda_1) + a_{11}(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]Tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{(\pi^2(-\pi - ((-1-i)\pi)) + (-\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi))Tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((\pi^2((-1+i)\pi + \pi)) + (-\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi))Tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(\pi^2((-1-i)\pi - (-1+i)\pi)) + (-\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2)Tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(((1-i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi))Tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{21} &= \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_3 - \lambda_2) + a_{21}(\lambda_2^2 - \lambda_3^2)]Tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad + \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_1 - \lambda_3) + a_{21}(\lambda_3^2 - \lambda_1^2)]Tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31})(\lambda_2 - \lambda_1) + a_{21}(\lambda_1^2 - \lambda_2^2)]Tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{((1-2i)\pi^2(-\pi - ((-1-i)\pi)) + (i\pi)((-1-i)\pi)^2 - \pi^2))Tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((1-2i)\pi^2((-1+i)\pi + \pi)) + (i\pi)(\pi^2 - ((-1+i)\pi)^2))Tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((1-2i)\pi^2((-1-i)\pi - (-1+i)\pi)) + (i\pi)((-1-i)\pi)^2 - ((-1-i)\pi)^2))Tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = iTanh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{31} = & \frac{[(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33})(\lambda_3 - \lambda_2) + a_{31}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_1 - \lambda_3) + a_{31}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{31} + a_{21}a_{31} + a_{31}a_{33})(\lambda_2 - \lambda_1) - a_{31}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{(0(-\pi - ((-1-i)\pi)) + (0)(((-1-i)\pi)^2 - \pi^2))\tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)(((-1+i)\pi)^2 - ((-1-i)\pi)^2))\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{12} = & \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_3 - \lambda_2) + a_{12}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_1 - \lambda_3) + a_{12}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& + \frac{[(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32})(\lambda_2 - \lambda_1) - a_{12}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& = \frac{(0(-\pi - ((-1-i)\pi)) + (0)(((-1-i)\pi)^2 - \pi^2))\tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)(((-1+i)\pi)^2 - ((-1-i)\pi)^2))\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{13} = & \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_3 - \lambda_2) + a_{13}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_1 - \lambda_3) + a_{13}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})(\lambda_2 - \lambda_1) - a_{13}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
= & \frac{((1+2i)\pi^2(-\pi - ((-1-i)\pi) + (-i\pi)((-1-i)\pi)^2 - \pi^2))\tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{((1+2i)\pi^2((-1+i)\pi + \pi) + (-i\pi)(\pi^2 - ((-1+i)\pi)^2))\tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{((1+2i)\pi^2((-1-i)\pi - (-1+i)\pi) + (-i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -i\tanh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{22} = & \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_3 - \lambda_2) + a_{22}(\lambda_2^2 - \lambda_3^2) + \lambda_2\lambda_3(\lambda_3 - \lambda_2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{22}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{21} + a_{22}^2 + a_{23}a_{32})(\lambda_2 - \lambda_1) - a_{22}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
= & \frac{(2i\pi^2(-\pi - ((-1-i)\pi) + ((-1-i)\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi))\tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(2i\pi^2((-1+i)\pi + \pi) + ((-1-i)\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi))\tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(2i\pi^2((-1-i)\pi - (-1+i)\pi) + ((-1-i)\pi)((-1+i)\pi)^2)\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& + \frac{(-((1-i)\pi)^2) + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi))\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = -i\tanh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{23} = & \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_3 - \lambda_2) + a_{23}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{32})(\lambda_1 - \lambda_3) + a_{23}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33})(\lambda_2 - \lambda_1) - a_{23}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{((1-6i)\pi^2(-\pi - ((-1-i)\pi) + (3i\pi)((-1-i)\pi)^2 - \pi^2))\tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1-6i)\pi^2((-1+i)\pi + \pi) + (3i\pi)(\pi^2 - ((-1+i)\pi)^2))\tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{((1-6i)\pi^2((-1-i)\pi - (-1+i)\pi) + (3i\pi)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 3i\tanh[\pi]
\end{aligned}$$

$$\begin{aligned}
p_{32} = & \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_3 - \lambda_2) + a_{32}(\lambda_2^2 - \lambda_3^2)]\tan[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_1 - \lambda_3) + a_{32}(\lambda_3^2 - \lambda_1^2)]\tan[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
& \frac{[(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33})(\lambda_2 - \lambda_1) - a_{32}(\lambda_2^2 - \lambda_1^2)]\tan[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
& \frac{(0(-\pi - ((-1-i)\pi) + (0)((-1-i)\pi)^2 - \pi^2))\tan[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(0((-1+i)\pi + \pi) + (0)(\pi^2 - ((-1+i)\pi)^2))\tan[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
& \frac{(0((-1-i)\pi - (-1+i)\pi) + (0)((-1+i)\pi)^2 - ((-1-i)\pi)^2))\tan[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = 0
\end{aligned}$$

$$\begin{aligned}
p_{33} &= \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_3 - \lambda_2) + a_{33}(\lambda_2^2 - \lambda_3^2) + (\lambda_3 - \lambda_2)\lambda_2\lambda_3] \operatorname{Tan}[\lambda_1]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_1 - \lambda_3) + a_{33}(\lambda_3^2 - \lambda_1^2) + \lambda_1\lambda_3(\lambda_1 - \lambda_3)] \operatorname{Tan}[\lambda_2]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} + \\
&\quad \frac{[(a_{13}a_{31} + a_{23}a_{32} + a_{33}^2)(\lambda_2 - \lambda_1) - a_{33}(\lambda_2^2 - \lambda_1^2) + \lambda_1\lambda_2(\lambda_2 - \lambda_1)] \operatorname{Tan}[\lambda_3]}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)} = \\
&= \frac{(-2i\pi^2(-\pi - ((-1-i)\pi) + ((-1+i)\pi)((-1-i)\pi)^2 - \pi^2) + (-(-1-i)\pi^2)(-\pi - (-1-i)\pi)) \operatorname{Tan}[(-1+i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{((-2i\pi^2((-1+i)\pi + \pi) + ((-1+i)\pi)(\pi^2 - ((-1+i)\pi)^2) + (-1+i)\pi(-\pi)((-1+i)\pi + \pi)) \operatorname{Tan}[(-1-i)\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(-2i\pi^2((-1-i)\pi - (-1+i)\pi) + ((-1+i)\pi)((-1+i)\pi)^2) \operatorname{Tan}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} + \\
&\quad + \frac{(-(-1-i)\pi)^2 + ((-1+i)\pi(-1-i)\pi)((-1-i)\pi - (-1+i)\pi)) \operatorname{Tan}[-\pi]}{(-\pi - (-1+i)\pi)(-\pi - (-1-i)\pi)((-1-i)\pi - (-1+i)\pi)} = i \operatorname{Tanh}[\pi]
\end{aligned}$$

Το αποτέλεσμά είναι ο πίνακας:

$$\operatorname{Tan}[A] = \begin{pmatrix} 0 & 0 & -i \operatorname{Tanh}[\pi] \\ -i \operatorname{Tanh}[\pi] & -i \operatorname{Tanh}[\pi] & 3i \operatorname{Tanh}[\pi] \\ 0 & 0 & i \operatorname{Tanh}[\pi] \end{pmatrix}$$

➤ Τέλος ο πίνακας $\operatorname{Cot}[A] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ δεν ορίζεται.

ΒΙΒΛΙΟΓΡΑΦΙΑ

- [1] **Μαρία Χ. Γουσίδου Κουτίτα** 2004, Αριθμητική Ανάλυση, Εκδόσεις Χριστοδούλιδη
- [2] **Μιχαήλ Ν. Βραχάτης** 2002, Αριθμητική Ανάλυση, Εκδόσεις Ελληνικά Γράμματα
- [3] **Jean-Paul Calvi** 2005, Lectures on multivariate polynomial interpolation
- [4] **Α. Χατζηδήμου** 1979, Αριθμητική Ανάλυση II, Πανεπιστήμιο Ιωαννίνων
- [5] **Σ. Μ. Μποζαπαλίδη**, Θεσσαλονίκη 1997, Εισαγωγή στην Άλγεβρα
- [6] **Π.Χ.Γ. Βασιλείου, Γ. Τσακλίδης**, Θεσσαλονίκη 2003, Εφαρμοσμένη Θεωρία Πινάκων, Εκδόσεις ΖΗΤΗ
- [7] **Ιωάννης Α. Τσαγκράκης**, Ηράκλειο 2009, Γραμμική Άλγεβρα
- [8] **Ανδρέας Λ. Πετράκης**, «Τριγωνομετρία Πινάκων» ένα νέο και ενδιαφέρον πεδίο για την έρευνα και την εκπαίδευση
- [9] **ΠΟΛΥΜΕΤΑΒΛΗΤΗ ΠΟΛΥΩΝΥΜΙΚΗ ΠΑΡΕΜΒΟΛΗ, ΑΛΕΞΑΝΔΡΟΣ ΕΥΡΙΠΙΔΟΥ**, Μεταπτυχιακή διπλωματική εργασία, Θεσσαλονίκη, Απρίλιος 2009
- [10] ΑΡΙΘΜΗΤΙΚΕΣ ΜΕΘΟΔΟΙ ΓΙΑ ΤΟΝ ΥΠΟΛΟΓΙΣΜΟ ΙΔΙΟΤΙΜΩΝ ΚΑΙ ΙΔΙΟΔΙΑΝΥΣΜΑΤΩΝ, Διπλωματική Εργασία, ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ, ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ, ΗΡΑΚΛΕΙΟ 2008